XIII. *On the Reflection and Refraction of Light.*

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Introduction.

The object of the following paper is to examine how far the hypothesis of a thin layer of transition between two transparent media will explain in detail the phenomena connected with the elliptic polarization produced by reflection at the boundary of two such media.

This problem has been approached by the following writers:—L. Lorenz, 'Poggendorff Annalen,' 114, p. 460; Van Ryn van Alkemaade, 'Wiedemann Annalen,' 20, p. 23; and P. Drude, 'Wiedemann Annalen,' 34 and 36.

Lorenz starts on the basis of the elastic solid theory, assuming that Fresnel's formulæ hold for a very small change of refractive index, and deduces expressions holding for a finite change of refrangibility, which are slight modifications of Fresnel's formulæ, and clearly unsound, since a rigid elastic solid theory must lead to Green's formulæ, and not to Fresnel's, as a first approximation. Fresnel's formulæ ought not without examination to be assumed to hold even for a very small change of refractive index, for the rate of change of refrangibility in crossing the boundary must be very rapid in order to produce a finite change, in a distance of the order of a wave-length.

Van Ryn van Alkemaade treats only of the electromagnetic theory of light—by successive approximation. His expressions for the change of phase are the same as in the following paper, namely (with notation changed from his),

$$\tan(\rho\perp) = 2\,\delta\mu_0 \cos i_0 \frac{\mu_1^2 - \mathrm{A} - (\mu_1^2\mathrm{G} - \mathrm{A})\sin^2 i_1}{\mu_1^2 - \mu_0^2} \quad \text{and} \quad \tan(\rho\,||) = 2\,\delta\mu_0 \cos i_0 \frac{\mu_1^2 - \mathrm{A}}{\mu_1^2 - \mu_0^2};$$

but for the amplitudes he gets

$$(\mathrm{R}\perp)^2 = \frac{\tan^2(i_0 - i_1)}{\tan^2(i_0 + i_1)}[1 + \tan^2(\rho\,||)], \quad (\mathrm{R}\,||)^2 = \frac{\sin^2(i_0 - i_1)}{\sin^2(i_0 + i_1)}[1 + \tan^2(\rho\,||)],$$

which are incomplete, taking no account of other terms of the same order involving B − C (see p. 849, *seq.*).

P. Drude treats the subject from the standpoint of Voigt's elastic solid theory, and obtains analogous formulæ. He uses Kirchhoff's boundary conditions, and since these are at best hypothetical, his method is not perfectly satisfactory.

In the following paper the employment of more or less hypothetical boundary conditions is avoided by supposing the medium continuous, the transition taking place in a variable layer of small but finite thickness, and solutions of the equations of vibration are obtained in ascending powers of the thickness, which expressions are at least as convergent as the geometric progression whose ratio is $\left(\frac{2\pi d}{\lambda}\bar{\mu}\right)^2$, where d is the thickness of the variable layer, $\bar{\mu}$ is the greatest value of the refractive index occurring in it, and λ is the wave-length of light. Expressions are then found for the intensities and phases of the reflected and refracted light, taking into account terms of order d^2.

The consequences are examined both of a rigid elastic solid theory, which includes the theories of Voigt and K. Pearson, and of the electromagnetic theory and Lord Kelvin's contractile ether theory, which lead to the same result.

The elastic solid theory gives modifications of Green's expressions, even when the refractive index of the pressural wave differs from that of light, and cannot be made to agree with experiment.

The electromagnetic and contractile ether theories lead to Cauchy's type of expression, the ellipticity being variable, and these agree very well with experiment.

§ 1. *General Equations of Vibration.*

It will be well briefly to recapitulate the systems of equations which have been proposed to represent the periodic disturbances to which light is due.

Electromagnetic Theory.—Let $\xi\epsilon^{-\iota pt}$, $\eta\epsilon^{-\iota pt}$, $\zeta\epsilon^{-\iota pt}$, $\lambda\epsilon^{-\iota pt}$, $\mu\epsilon^{-\iota pt}$, $\nu\epsilon^{-\iota pt}$ represent the components of electric and magnetic force for a periodic disturbance at the point (xyz) of the medium, where its specific inductive capacity is K—the real parts of the complex expressions being taken in the usual way. Also let the velocity of propagation of electromagnetic disturbance *in vacuo* be 1/A. Then the equations of vibration are

$$A\iota p\,.\,\lambda = \frac{\partial\zeta}{\partial y} - \frac{\partial\eta}{\partial z}\,, \quad A\iota p\,.\,K\xi = \frac{\partial\mu}{\partial z} - \frac{\partial\nu}{\partial y} \text{ (and two similar pairs),}$$

whence

$$\frac{\partial}{\partial y}\left(\frac{\partial\xi}{\partial y} - \frac{\partial\eta}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\xi}{\partial z} - \frac{\partial\zeta}{\partial x}\right) + A^2Kp^2\xi = 0 \text{ (and two others)} \quad . \quad . \quad \text{(I.).}$$

These are given by Hertz ("Ueber die Grundgleichungen der Elektrodynamik," 'WIED. Ann.,' 40), as according with experiment for heterogeneous media in the absence of free electricity.

But following Lord RAYLEIGH ("Electromagnetic Theory of Light," 'Phil. Mag.,' 1881), I have put the magnetic permeability = 1 in HERTZ's equations, so as to make them give results agreeing with experiments on reflection of light and on the scattering of light by small particles. There are also electrical experiments to justify this course, due to HERTZ, and showing that the phenomena of, at any rate, quick vibrations, are independent of the magnetic permeability of the medium.

Elastic Solid Theory.—Let $u\epsilon^{-\iota pt}$, $v\epsilon^{-\iota pt}$, $w\epsilon^{-\iota pt}$ represent components of displacement at the point (xyz) of the medium, where the effective density is ρ, the rigidity is n, and the bulk-modulus is k. Following Lord RAYLEIGH ("On the Scattering of Light by Small Particles," 'Phil. Mag.,' 1871), we shall suppose n the same in all bodies, and therefore constant throughout the variable medium considered. Then the equations of vibration in LAMÉ's form are

$$\frac{\partial}{\partial x}\left\{\left(k+\tfrac{4}{3}n\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right\}+n\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+n\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$
$$+\rho p^2 u=0 \text{ (and two others)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II.)}.$$

These equations include the results of the more general theories of VOIGT and of K. PEARSON.

VOIGT ("Theorie des Lichtes für durchsichtige Medien," 'WIED. Ann.,' 19, p. 873) neglects the first pressural term, and replaces n, ρp^2 respectively by $e + a - a'p^2$ and $(m+r)p^2 - n$, that is, makes the effective density and rigidity depend on the period.

K. PEARSON ("Generalized Equations of Elasticity," 'Proc. Lond. Math. Soc.', vol. 20, p. 291) replaces $k+\frac{4}{3}n$, n, and ρp^2 by $\lambda+2\mu+(\lambda'+2\mu')p^2$, $\mu+(\mu'+\frac{1}{4}\gamma)p^2$, and by $(\rho-\kappa)p^2$.

Thus, in the general case, k, n, ρ are functions of the period.

There are two principal forms of elastic solid theory—

First.—GREEN's *Theory*—which attempts to get rid of the longitudinal (pressural) waves by a kind of total reflection at all but very small angles of incidence, whilst at nearly normal incidence their effect is inappreciable owing to the smallness of the normal component.

The bulk-modulus k is made very large, the expansion $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ is very small, the pressure is finite—it is not necessary that k be greater than $100n$ to make the effect of the longitudinal wave inappreciable. (GLAZEBROOK, 'B.A. Report on Optics,' 1885, p. 192.)

Secondly.—Lord KELVIN's *Contractile Ether Theory*—$k+\frac{4}{3}n$ is made zero, so that the longitudinal wave is not propagated from any place where it may arise. Putting zero for $k+\frac{4}{3}n$ in equations (II.), they become of exactly the same form as equations (I.) for the electromagnetic theory. These two theories will, therefore, be considered together.

We shall suppose the medium in which the disturbances take place to be perfectly continuous, though its qualities may vary from place to place. It follows that ξ, η, ζ, u, v, w must be continuous functions of (xyz), as well as their first differential coefficients, and this condition must replace boundary conditions at places where the nature of the medium changes, however rapidly it may do so.

§ 2. *Waves in a Variable Layer between two Media.*

For our purpose it is only necessary to consider the very special case when the heterogeneous medium is arranged in plane layers, perpendicular to Ox suppose, and we shall further suppose the variable portion to be a thin layer separating two media of different but constant quality, into each of which the layer passes continuously.

We shall suppose plane waves incident in the first medium, which will give rise to plane reflected and refracted waves. Take Oz perpendicular to the plane of incidence; then Oy will be parallel to the intersections of the plane of incidence with the plane layers, and since the traces of all the waves on the plane layers must move along these layers at the same rate, the coefficient of y must be the same in the expressions for the different waves.

Let now λ be the wave-length in vacuo of the light $\epsilon^{-\iota pt}$, μ its refractive index from vacuum into the variable layer, μ_0, μ_1 the values for the two media on either side. Then we have $\frac{\rho p^2}{n}$ (or A^2Kp^2) $= \frac{4\pi^2}{\lambda^2}\mu^2$. If i be the angle the wave-normal makes with Ox, the coefficient of y in the expression of the wave will be $\frac{2\pi}{\lambda}\mu \sin i$, which, being everywhere the same, we shall write $\frac{2\pi}{\lambda}\nu$.

Also write $\frac{k + \frac{4}{3}n}{n} = m^2$, so that the velocity of the pressural wave is m times that of the transverse wave.

By what has been said above, μ, m are functions of x only.

And the displacements, &c., are independent of z, and proportional to $\epsilon^{\iota\left(\frac{2\pi\nu}{\lambda}y - pt\right)}$.

First medium.

This is of constant quality (μ_0, m_0) and extends from $x = -\infty$ to $x = 0$; the equations (II.) become

$$\left.\begin{aligned} m_0^2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{4\pi^2}{\lambda^2}\mu_0^2 . u &= 0 \\ m_0^2\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{4\pi^2}{\lambda^2} . \mu_0^2 . v &= 0 \end{aligned}\right\} \text{Vibrations parallel to plane of incidence.}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{4\pi^2}{\lambda^2} \,.\, \mu_0^2 \,.\, w = 0 \quad \text{Vibrations perpendicular to plane of incidence.}$$

These are satisfied by

$$u = \left(- \sin i_0 \,.\, \epsilon^{\iota \frac{2\pi\mu_0 \cos i_0}{\lambda} \,.\, x} + r \sin i_0 \,.\, \epsilon^{-\iota \frac{2\pi\mu_0 \cos i_0}{\lambda} x} + r'\alpha_0 \,.\, \epsilon^{\frac{2\pi\mu_0\alpha_0}{\lambda} x}\right) . \, \epsilon^{\iota \left(\frac{2\pi\nu}{\lambda} y - pt\right)}$$

$$v = \left(\cos i_0 \,.\, \epsilon^{\iota \frac{2\pi\mu_0 \cos i_0}{\lambda} x} + r \cos i_0 \,.\, \epsilon^{-\iota \frac{2\pi\mu_0 \cos i_0}{\lambda} x} + \iota r' \sin i_0 \,.\, \epsilon^{\frac{2\pi\mu_0\alpha_0}{\lambda} x}\right) . \, \epsilon^{\iota \left(\frac{2\pi\nu}{\lambda} y - pt\right)}$$

$$w = \left(\epsilon^{\iota \frac{2\pi\mu_0 \cos i_0}{\lambda} x} + r\epsilon^{-\iota \frac{2\pi\mu_0 \cos i_0}{\lambda} x}\right) . \, \epsilon^{\iota \left(\frac{2\pi\nu}{\lambda} y - pt\right)} .$$

Here the first term represents an incident wave, the second the reflected, and the third a pressural wave, which last travels along the boundary $x = 0$, and rapidly diminishes away from that boundary.

i_0 is the angle of incidence; α_0 is a constant which is found to be $+ \sqrt{\left(\sin^2 i_0 - \frac{1}{m_0^2}\right)}$. r, r' are complex constants $\mathrm{R}\epsilon^{\iota\rho}$, $\mathrm{R}'\epsilon^{\iota\rho'}$; R, R$'$ are the amplitudes, ρ, ρ' the retardations of phase, of the reflected and pressural waves.

The pressure is proportional to $m_0^2 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = - \frac{2\pi\mu_0}{\lambda} \,.\, r' \,.\, \epsilon^{\frac{2\pi\mu_0\alpha_0}{\lambda} x + \iota \left(\frac{2\pi\nu}{\lambda} y - pt\right)}$; thus r' vanishes for the electromagnetic theory and for Lord KELVIN's theory, for which m_0 vanishes.

Second Medium.

This is of constant quality (μ_1, m_1) and extends from $x = d$ to $x = \infty$; the equations (II.) are

$$\left.\begin{aligned} m_1^2 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{4\pi^2}{\lambda^2} \,.\, \mu_1^2 \,.\, u = 0 \\ m_1^2 \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{4\pi^2}{\lambda^2} \,.\, \mu_1^2 \,.\, v = 0 \end{aligned}\right\} \text{Vibrations parallel to plane of incidence.}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{4\pi^2}{\lambda^2} \,.\, \mu_1^2 \,.\, w = 0 \quad \text{Vibrations perpendicular to plane of incidence.}$$

They are satisfied by

$$u = - \left(s \sin i_1 \,.\, \epsilon^{\iota \frac{2\pi\mu_1 \cos i_1}{\lambda} (x - d)} + s'\alpha_1 \,.\, \epsilon^{- \frac{2\pi\mu_1\alpha_1}{\lambda} (x - d)}\right) . \, \epsilon^{\iota \left(\frac{2\pi\nu}{\lambda} y - pt\right)}$$

$$v = \left(s \cos i_1 . \, \epsilon^{\iota \frac{2\pi\mu_1 \cos i_1}{\lambda} (x - d)} + \iota \, s' \sin i_1 . \, \epsilon^{\iota - \frac{2\pi\mu_1\alpha_1}{\lambda} (x - d)}\right) . \, \epsilon^{\iota \left(\frac{2\pi\nu}{\lambda} y - pt\right)}$$

$$w = s \,.\, \epsilon^{\iota \left(\frac{2\pi\mu_1 \cos i_1}{\lambda} (x - d) + \frac{2\pi\nu}{\lambda} y - pt\right)} .$$

The first term represents the refracted wave, the second the pressural wave, which only exists close to the boundary $x = d$.

i_1 is the angle of refraction, which obeys SNELL'S law, $\mu_1 \sin i_1 = \nu = \mu_0 \sin i_0$.

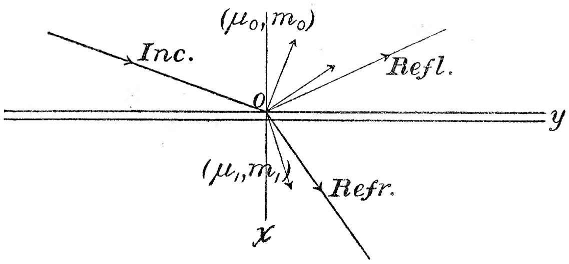
α_1 is a constant found to be $+\sqrt{\left(\sin^2 i_1 - \frac{1}{m_1^2}\right)}$.

s, s' are complex constants $S\epsilon^{\iota\sigma}$, $S'\epsilon^{\iota\sigma'}$; S, S′ are the amplitudes, σ, σ' the retardations of phase, of the refracted and pressural waves.

The pressure is proportional to

$$m_1^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{2\pi\mu_1}{\lambda} \cdot s' . \epsilon^{-\frac{2\pi\mu_1\alpha_1}{\lambda}(x-d) + \iota\left(\frac{2\pi\nu}{\lambda}y - pt\right)}.$$

The signs of R, S are chosen so that at normal incidence v shall be + for each wave, as shown in the figure, when the signs of R, S are +.



Variable Layer.

It extends from $x = 0$ to $x = d$ and is continuous with the media bounding it. The displacements and their first differential coefficients with respect to x must have the same values at the boundaries in the variable layer and in the media beyond, giving in all twelve boundary conditions, six of which determine the motion in the variable layer, and the remaining six determine the constants r, r', s, s' for vibrations parallel and perpendicular to the plane of incidence.

We may write the displacements in the form $u . \epsilon^{\iota\left(\frac{2\pi\nu}{\lambda}y - pt\right)}$; write also

$$m^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{2\pi}{\lambda} \cdot \Pi . \epsilon^{\iota\left(\frac{2\pi\nu}{\lambda}y - pt\right)};$$

then $u \,..\, \Pi$ are functions of x alone, and Π is proportional to the pressure, which vanishes for those theories which make m zero.

The equations (II.) become

$$\left.\begin{aligned} -\frac{2\pi}{\lambda} \cdot \frac{d\Pi}{dx} - \iota\frac{2\pi\nu}{\lambda}\frac{dv}{dx} + \frac{4\pi^2}{\lambda^2}(\mu^2 - \nu^2)\,u &= 0 \\ -\iota\frac{4\pi^2}{\lambda^2} \cdot \nu\Pi - \iota\frac{2\pi\nu}{\lambda} \cdot \frac{du}{dx} + \frac{d^2v}{dx^2} + \frac{4\pi^2}{\lambda^2} \cdot \mu^2 . v &= 0 \\ m^2\left(\frac{du}{dx} + \iota\frac{2\pi\nu}{\lambda}v\right) + \frac{2\pi}{\lambda} . \Pi &= 0 \end{aligned}\right\} \text{Vibrations parallel to plane of incidence.}$$

$$\frac{d^2w}{dx^2} + \frac{4\pi^2}{\lambda^2} \cdot (\mu^2 - \nu^2) \,.\, w = 0 \quad \text{Vibrations perpendicular to plane of incidence.}$$

We shall choose u, v, Π, w, so that, when $x = 0$, $u = u_0$, $v = v_0$, $\Pi = \Pi_0$, $w = w_0$, and when $x = d$, $v = v_1$, $w = w_1$, where

$$u_0 = -\sin i_0 (1 - r) + \alpha_0 r', \quad v_0 = \cos i_0 (1 + r) + \iota \sin i_0 \,.\, r', \quad \Pi_0 = \mu_0 r', \quad w_0 = 1 + r,$$
$$v_1 = \cos i_1 \,.\, s + \iota \sin i_1 \,.\, s', \quad w_1 = s.$$

The six conditions determining r, r' . . . will be

When $x = 0$, $\dfrac{dv}{dx} = \iota \dfrac{2\pi\mu_0}{\lambda} \{\cos^2 i_0 \,.\, (1 - r) + \alpha_0 \sin i_0 \,.\, r'\}$, $\dfrac{dw}{dx} = \iota \dfrac{2\pi\mu_0 \cos i_0}{\lambda} (1 - r)$.

When $x = d$: $u = -(\sin i_1 \,.\, s + \alpha_1 s')$, $\Pi_1 = \mu_1 s'$, $\dfrac{dv}{dx} = \iota \dfrac{2\pi\mu_1}{\lambda} \,.\, \{\cos^2 i_1 . s - \alpha_1 \sin i_1 . s'\}$,

$$\frac{dw}{dx} = \iota \frac{2\pi\mu_1 \cos i_1}{\lambda} \,.\, s.$$

§ 3. *Determination of the Displacements for the Variable Layer.*

It is in general impossible to solve the equations in finite terms; in the physical problem the transition layer may be considered thin even in comparison with the wave-length, and the equations can be solved in very convergent series, proceeding in ascending powers of some small quantity depending on the thickness of the variable layer. This quantity we shall take to be $\delta \equiv 2\pi d/\lambda$. Putting also ξ for x/d, the value of ξ will lie between 0 and 1; and the equations become

$$\left.\begin{aligned} \frac{d\Pi}{d\xi} + \iota\nu \,.\, \frac{dv}{d\xi} - \delta(\mu^2 - \nu^2) \,.\, u = 0 \\ \frac{d^2v}{d\xi^2} - \iota\delta\nu\frac{du}{d\xi} - \iota\delta^2 \,.\, \nu\Pi + \delta^2\mu^2 v = 0 \\ m^2\frac{du}{d\xi} + \iota\delta m^2\nu \,.\, v + \delta\Pi = 0 \end{aligned}\right\} \begin{array}{l} \text{Vibrations parallel to plane of} \\ \quad \text{incidence. . . . . . .} \quad \text{(III.),} \end{array}$$

$$\frac{d^2w}{d\xi^2} + \delta^2 \,.\, (\mu^2 - \nu^2) w = 0 \quad \text{Vibrations perpendicular to plane of incidence} \quad \text{(IV.),}$$

and when $\xi = 0$, $u = u_0$, $v = v_0$, $\Pi = \Pi_0$, $w = w_0$; and when $\xi = 1$, $v = v_1$, $w = w_1$.

It will be necessary to treat separately the cases of the electromagnetic and contractile ether theories on the one hand, for which m, Π are zero, and of the elastic solid theory on the other hand, for which m is very large and Π is finite. In this latter case we shall neglect $1/m^2$ in the small terms, which are themselves only corrections due to the finite, though small, thickness of the transition layer.

Vibrations perpendicular to the plane of incidence (all theories).

The equation to be solved is (IV.), p. 829, viz.:—$d^2w/d\xi^2 + \delta^2 . (\mu^2 - \nu^2)\, w = 0$, with the conditions that when $\xi = 0$, $w = w_0$, when $\xi = 1$, $w = w_1$.

Put $w = w^0 + \delta^2 w^1 + \ldots$, where $d^2w^0/d\xi^2 = 0$, $d^2w^1/d\xi^2 + (\mu^2 - \nu^2)\, w^0 = 0, \ldots$ and when $\xi = 0$, $w^0 = w_0$, $w^1 = \ldots = 0$, when $\xi = 1$, $w^0 = w_1$, $w^1 = \ldots = 0$.

These give

$$w^0 = w_0(1-\xi) + w_1 . \xi,\quad w^1 = -\int_0^\xi (\mu^2 - \nu^2) w^0 . (\xi - \eta)\, d\eta + \xi . \int_0^1 (\mu^2 - \nu^2) w^0 . (1-\eta)\, d\eta, \ldots$$

whence

$$dw^0/d\xi = -w_0 + w_1,\quad dw^1/d\xi = -\int_0^\xi (\mu^2 - \nu^2) w^0 . d\eta + \int_0^1 (\mu^2 - \nu^2) . w^0 . (1-\eta)\, d\eta, \ldots$$

Let a bar – written over a quantity denote its greatest numerical value between $\xi = 0$ and $\xi = 1$, *e.g.*, $\bar{\mu}$ the maximum refractive index.

$\overline{w^{(n)}}$ is given by $dw^{(n)}/d\xi = 0$, or by

$$\int_0^1 (\mu^2 - \nu^2) . w^{(n-1)} . (1-\eta)\, d\eta = \int_0^\xi (\mu^2 - \nu^2)\, w^{(n-1)} . d\eta .$$

Hence $\overline{w^{(n)}} = \int_0^\xi (\mu^2 - \nu^2)\, w^{(n-1)} . \eta\, d\eta$ where ξ lies between 0 and 1, and therefore $\overline{w^{(n)}} < \frac{1}{2} \overline{w^{(n-1)}} . \overline{(\mu^2 - \nu^2)}$. Now $\nu^2 = \mu_0{}^2 \sin^2 i_0 < \mu_0{}^2 < \bar{\mu}^2$, and therefore $\overline{(\mu^2 - \nu^2)} < \bar{\mu}^2$, and $\overline{w^{(n)}} < \frac{1}{2} \bar{\mu}^2 . \overline{w^{(n-1)}}$. It follows that $w < \overline{w^0} . \{1 + \frac{1}{2}\delta^2 . \bar{\mu}^2 + \frac{1}{4}\delta^4 . \bar{\mu}^4 + \ldots\} < \frac{\overline{w^0}}{1 - \frac{1}{2}\delta^2 \bar{\mu}^2}$, which is finite as long as $\delta < \sqrt{\frac{2}{\bar{\mu}^2}}$ or $\frac{d}{\lambda} < \frac{1}{4{\cdot}53 . \bar{\mu}}$. We shall neglect powers of δ above the second, so that we may write $w = w^0 + \delta^2 . w^1$.

Then $dw^0/d\xi = w_1 - w_0$.

$$\left(\frac{dw^1}{d\xi}\right)_{\xi=0} = \int_0^1 (\mu^2 - \nu^2)\, w^0 . (1-\eta)\, d\eta$$

$$= w_0 . \int_0^1 (\mu^2 - \nu^2)(1-\eta)^2\, d\eta + w_1 \int_0^1 (\mu^2 - \nu^2)\, \eta\, (1-\eta)\, d\eta$$

$$= w_0 \left\{ \int_0^1 \mu^2 . (1-\eta)^2\, d\eta - \frac{\nu^2}{3} \right\} + w_1 \left\{ \int_0^1 \mu^2 . \eta\, (1-\eta)\, d\eta - \frac{\nu^2}{6} \right\}.$$

$$\left(\frac{dw^1}{d\xi}\right)_{\xi=1} = -\int_0^1 (\mu^2 = \nu^2)\, w^0 . \eta\, d\eta$$

$$= -\, w^0 . \left\{ \int_0^1 \mu^2 . \eta\, (1-\eta)\, d\eta - \frac{\nu^2}{6} \right\} - w_1 . \left\{ \int_0^1 \mu^2 . \eta^2 . d\eta - \frac{\nu^2}{3} \right\} .$$

Hence

$$\left.\begin{aligned}\left(\frac{dw}{d\xi}\right)_0 &= -w_0 . \left[1 - \delta^2 . \left\{\int_0^1 \mu^2 (1-\eta)^2 \, d\eta - \tfrac{1}{3}\nu^2\right\}\right] \\ &\quad + w_1 . \left[1 + \delta^2 . \left\{\int_0^1 \mu^2 . \eta \, (1-\eta) \, d\eta - \tfrac{1}{6}\nu^2\right\}\right] \\ \left(\frac{dw}{d\xi}\right)_1 &= -w_0 . \left[1 + \delta^2 . \left\{\int_0^1 \mu^2 . \eta \, (1-\eta) \, d\eta - \tfrac{1}{6}\nu^2\right\}\right] \\ &\quad + w_1 . \left[1 - \delta^2 . \left\{\int_0^1 \mu^2 \eta^2 \, d\eta - \tfrac{1}{3}\nu^2\right\}\right]\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad \text{(V.)}.$$

Vibrations parallel to plane of incidence (Electromagnetic and Contractile Ether Theories).

The equations to be solved are (III.), p. 829, with m, Π zero, viz.,

$$\iota \nu \frac{dv}{d\xi} - \delta \left(\mu^2 - \nu^2\right) u = 0, \quad \frac{d^2 v}{d\xi^2} - \iota \, \delta \nu \frac{du}{d\xi} + \delta^2 . \mu^2 v = 0,$$

whence eliminating u

$$\frac{d}{d\xi}\left(\frac{\mu^2}{\mu^2 - \nu^2} \frac{dv}{d\xi}\right) + \delta^2 \mu^2 v = 0,$$

with the conditions that when $\xi = 0$, $v = v_0$, and when $\xi = 1$, $v = v_1$.

Put $v = v^0 + \delta^2 v' + \ldots$ where

$$\frac{d}{d\xi}\left(\frac{\mu^2}{\mu^2 - \nu^2} \frac{dv^0}{d\xi}\right) = 0, \quad \frac{d}{d\xi}\left(\frac{\mu^2}{\mu^2 - \nu^2} \cdot \frac{dv'}{d\xi}\right) = -\mu^2 v^0, \ldots$$

and when $\xi = 0$, $v^0 = v_0$, $v' = \ldots = 0$, when $\xi = 1$, $v^0 = v_1$, $v' = \ldots = 0$.

Write

$$\pi(\xi) = \int_0^\xi \frac{\mu^2 - \nu^2}{\mu^2} \, d\eta = \xi - \nu^2 . \int_0^\xi \frac{d\eta}{\mu^2},$$

so that $\pi(\xi)$ increases as long as $\nu^2 < \mu^2$ and maximum of $\pi < 1$. We have at once

$$v^0 = v_0 \frac{\pi(1) - \pi(\xi)}{\pi(1)} + v_1 \frac{\pi(\xi)}{\pi(1)},$$

$$v' = -\int_0^\xi \mu^2 . \{\pi(\xi) - \pi(\eta)\} . v^0 \, d\eta + \frac{\pi(\xi)}{\pi(1)} \int^1 \mu^2 \{\pi(1) - \pi(\eta)\} \, v_0 \, d\eta, \ldots$$

whence

$$\frac{\mu^2}{\mu^2 - \nu^2} \cdot \frac{dv^0}{d\xi} = \frac{v_1 - v_0}{\pi(1)}, \quad \frac{\mu^2}{\mu^2 - \nu^2} \cdot \frac{dv'}{d\xi} = -\int_0^\xi \mu^2 . v^0 . (\quad + \int_0^1 \mu^2 . \frac{\pi(1) - \pi(\eta)}{\pi(1)} \cdot v^0 . d\eta, \ldots$$

with the same notation as before

$$\overline{v}^{(n)} = \int_0^\xi \mu^2 \pi(\eta) . v^{(n-1)} . d\eta, \quad 0 < \xi < 1,$$

hence

$$\overline{v}^{(n)} < \overline{u}^2 \,.\, \overline{v}^{(n-1)}, \quad \text{and} \quad \overline{v} < \overline{v}^0 .\, \{1 + \delta^2\overline{\mu}^2 + \delta^4\overline{\mu}^4 + \ldots\} < \frac{\overline{v}^0}{1 - \delta^2\overline{\mu}^2},$$

which is finite as long as

$$\delta < \frac{1}{\mu} \quad \text{or} \quad \frac{d}{\lambda} < \frac{1}{6{\cdot}28 \,.\, \mu}.$$

As before we neglect δ^4 and higher powers, and find

$$\left(\frac{dv^0}{d\xi}\right)_{\xi=0} = \frac{\cos^2 i_0}{\pi(1)}(v_1 - v_0), \qquad \left(\frac{dv^0}{d\xi}\right)_{\xi=1} = \frac{\cos^2 i_1}{\pi(1)}(v_1 - v_0),$$

since $\nu = \mu \sin i$, and thus $\dfrac{\mu^2 - \nu^2}{\mu^2} = \cos^2 i$.

$$\begin{aligned}\left(\frac{dv'}{d\xi}\right)_{\xi=0} &= \frac{\cos^2 i_0}{\pi(1)} \int_0^1 \mu^2 .\, \{\pi(1) - \pi(\eta)\}\, v^0\, d\eta \\ &= \frac{\cos^2 i_0}{\{\pi(1)\}^2}\left[v_0 \int_0^1 \mu^2 .\, \{\pi(1) - \pi(\eta)\}^2 .\, d\eta + v_1 \int_0^1 \mu^2 .\, \{\pi(1) - \pi(\eta)\}\, \pi(\eta)\, d\eta\right]\end{aligned}$$

$$\begin{aligned}\left(\frac{dv'}{d\xi}\right)_{\xi=1} &= -\frac{\cos^2 i_1}{\pi(1)} \int_0^1 \mu^2 \pi(\eta) .\, v^0 .\, d\eta \\ &= -\frac{\cos^2 i}{\{\pi(1)\}^2}\left[v_0 \int_0^1 \mu^2 .\, \{\pi(1) - \pi(\eta)\}\, \pi(\eta)\, d\eta + v_1 \int_0^1 \mu^2 .\, \{\pi(\eta)\}^2 .\, d\eta\right].\end{aligned}$$

Hence

$$\left.\begin{aligned}\left(\frac{dv}{d\xi}\right)_{\xi=0} &= -\frac{\cos^2 i_0}{\pi(1)} v_0 \cdot \left[1 - \frac{\delta^2}{\pi(1)} \int_0^1 \mu^2 .\{\pi(1) - \pi(\eta)\}^2 d\eta\right] \\ &\quad + \frac{\cos^2 i_0}{\pi(1)} v_1 \cdot \left[1 + \frac{\delta^2}{\pi(1)} \int_0^1 \mu^2 .\, \{\pi(1) - \pi(\eta)\}\, \pi(\eta)\, d\eta\right] \\ \left(\frac{dv}{d\xi}\right)_{\xi=1} &= -\frac{\cos^2 i_0}{\pi(1)} v_0 \cdot \left[1 + \frac{\delta^2}{\pi(1)} \int_0^1 \mu^2 \{\pi(1) - \pi(\eta)\}\, \pi(\eta)\, d\eta\right] \\ &\quad + \frac{\cos^2 i_1}{\pi(1)} v_1 \cdot \left[1 - \frac{\delta^2}{\pi(1)} \int_0^1 \mu^2 .\, \{\pi(\eta)\}^2 d\eta\right]\end{aligned}\right\} \quad . \;.\; . \qquad \text{(VI.)}$$

Vibrations parallel to the plane of incidence (*Elastic Solid Theory*).

The equations to be solved are (III.), p. 829, adding the third multiplied by $\iota\delta\nu$ to the second for a new second,—

$$\frac{d\Pi}{d\xi} + \iota\nu\frac{dv}{d\xi} - \delta(\mu^2 - \nu^2)\, u = 0, \qquad \frac{d^2v}{d\xi^2} - \iota\delta^2\nu .\left(1 - \frac{1}{m^2}\right)\Pi + \delta^2 .\, (\mu^2 - \nu^2)\, v = 0,$$

$$\frac{du}{d\xi} + \iota\delta\nu \,.\, v + \delta\frac{\Pi}{m^2} = 0,$$

with the conditions that when $\xi = 0$, $u = u_0$, $v = v_0$, $\Pi = \Pi_0$, and when $\xi = 1$, $v = v_1$.

Put $u = u^0 + \delta u' + \delta^2 u'' + \ldots$, $v = v_0 + \delta v' + \delta^2 v'' + \ldots$, $\Pi = \Pi^0 + \delta\Pi' + \delta^2\Pi'' + \ldots$, where

$$\frac{du^0}{d\xi} = 0, \qquad \frac{d^2v^0}{d\xi^2} = 0,$$

$$\frac{du'}{d\xi} + \iota\nu v^0 + \frac{\Pi^0}{m^2} = 0, \qquad \frac{d^2v'}{d\xi^2} = 0,$$

$$\frac{du''}{d\xi} + \iota\nu v' + \frac{\Pi'}{m^2} = 0, \qquad \frac{d^2v''}{d\xi^2} - \iota\nu\left(1 - \frac{1}{m^2}\right)\Pi^0 + (\mu^2 - \nu^2)\, v^0 = 0,$$

$$\frac{d\Pi^0}{d\xi} + \iota\nu\frac{dv^0}{d\xi} = 0,$$

$$\frac{d\Pi'}{d\xi} + \iota\nu\frac{dv'}{d\xi} - (\mu^2 - \nu^2)\, u^0 = 0,$$

$$\frac{d\Pi''}{d\xi} + \iota\nu\frac{dv''}{d\xi} - (\mu^2 - \nu^2)\, u' = 0,$$

and when $\xi = 0$, $u^0 = u_0$, $v^0 = v_0$, $\Pi^0 = \Pi_0$, $u' = v' = \Pi' = \ldots = 0$, when $\xi = 1$, $v^0 = v_1$, $v' = \ldots = 0$.

We have at once

$$u^0 = u_0,\ v^0 = v_0(1 - \xi) + v_1\xi,\ \Pi^0 = \Pi_0 - \iota\nu\,.\,(v_1 - v_0)\,\xi,$$

$$u' = -\iota\nu\int_0^\xi v^0\,.\,d\eta - \int_0^\xi \frac{\Pi^0}{m^2}\,d\eta = -\iota\nu v_0\xi\left(1 - \tfrac{1}{2}\xi\right) - \tfrac{1}{2}\iota\nu\,.\,v_1\xi^2$$

$$+ \iota\nu(v_1 - v_0)\int_0^\xi \frac{\eta}{m^2}\,d\eta - \Pi_0\int_0^\xi \frac{d\eta}{m^2}.$$

$$v' = 0.$$

$$\Pi' = \int_0^\xi (\mu^2 - \nu^2)\, u^0\, d\eta = u_0\,.\left\{\int_0^\xi \mu^2\, d\eta - \nu^2\xi\right\}.$$

$$u'' = -\iota\nu\int_0^\xi v'\,d\eta - \int_0^\xi \frac{\Pi'\,d\eta}{m^2} = -\int_0^\xi \frac{\Pi'}{m^2}\,.\,d\eta.$$

$$v'' = \iota\nu\int_0^\xi\left(1 - \frac{1}{m^2}\right)\Pi^0\,.\,(\xi - \eta)\,d\eta - \int_0^\xi(\mu^2 - \nu^2)\,.\,v^0\,.\,(\xi - \eta)\,d\eta$$

$$- \iota\nu\xi\,.\int_0^1\left(1 - \frac{1}{m^2}\right)\Pi^0\,.\,(1 - \eta)\,d\eta + \xi\int_0^1(\mu^2 - \nu^2)\,v^0\,.\,(1 - \eta)\,d\eta.$$

$$\Pi'' = -\iota\nu v'' + \int_0^\xi(\mu^2 - \nu^2)\,u'\,d\eta.$$

In the same way as before denoting maximum value of v by $\bar{v}$ . . . we have,

$$\overline{u}^{(n)} < \nu\overline{v}^{(n-1)} + \overline{(\Pi^{(n-1)}/m^2)}, \qquad \overline{\Pi}^{(n)} < \nu\overline{v}^{(n)} + \overline{(\mu^2 - \nu^2)} \,.\, \overline{u}^{(n-1)},$$

$$\overline{v}^{(n)} < \tfrac{1}{2}\,\nu\overline{\Pi}^{(n-2)} + \tfrac{1}{2}\,\overline{(\mu^2 - \nu^2)} \,.\, \overline{v}^{(n-2)},$$

and since $\overline{v} < \overline{v}^0 + \delta\overline{v}' + \ldots$

$$\overline{u} < u_0 + \nu\,\delta\overline{v} + \delta \,.\, \overline{(1/m^2)} \,.\, \overline{\Pi}, \qquad \overline{v} < \overline{v}^0 + \tfrac{1}{2}\nu\,\delta^2\overline{\Pi} + \tfrac{1}{2}\,\overline{(\mu^2 - \nu^2)} \,.\, \delta^2 \,.\, \overline{v},$$

$$\overline{\Pi} < \nu\overline{v} + \delta \,.\, \overline{(\mu^2 - \nu^2)} \,.\, \overline{u},$$

or if $0 < \epsilon < 1$, $0 < \epsilon' < 1$,

$$\overline{u} \,.\, \{1 - \delta^2 \,.\, \overline{(\mu^2 - \nu^2)}\} - \nu\delta \,.\, \overline{v} \,.\, \{1 + \overline{(1/m^2)}\} = \epsilon u_0,$$

$$- \tfrac{1}{2}\nu\,\delta^3 \,.\, \overline{u}\,\overline{(\mu^2 - \nu^2)} + \overline{v} \,.\, \{1 - \tfrac{1}{2}\overline{\mu}^2 \,.\, \delta^2\} = \epsilon'\overline{v}^0,$$

whence

$$\overline{u} = \frac{\epsilon u_0 \,.\, (1 - \tfrac{1}{2}\delta^2\overline{\mu}^2) + \epsilon' \,.\, \overline{v}^0 \,.\, \nu\delta \,.\, \{1 + \overline{(1/m^2)}\}}{\{1 - \delta^2 . \overline{(\mu^2 - \nu^2)}\}\,\{1 - \tfrac{1}{2}\delta^2 . \overline{\mu}^2\} - \tfrac{1}{2}\nu^2 . \delta^4 . \{1 + \overline{(1/m^2)}\}\,\overline{(\mu^2 - \nu^2)}},$$

and this is finite, and so also are $\overline{v}$, $\overline{\Pi}$, as long as the denominator does not vanish. Writing this denominator in the form $(1 - \frac{1}{2}\alpha\delta^2)\,(1 - \frac{1}{2}\beta\delta^2)$, we have

$$\alpha + \beta = 2\,\overline{(\mu^2 - \nu^2)} + \overline{\mu}^2, \quad \alpha\beta = \tfrac{1}{2}\,\overline{(\mu^2 - \nu^2)}\,\{\overline{\mu^2 - \nu^2} - \overline{\nu^2/m^2}\}.$$

Now m is large, at least 10, and $\overline{\mu^2 - \nu^2}$ is at least ${\mu_0}^2 \cos^2 i_0$, hence β is + or very small negative, in which latter case $\overline{\mu^2 - \nu^2}$ is small; thus, α, the larger of the two, $< 3\overline{\mu}^2$. Hence $\overline{u}$, $\overline{v}$, $\overline{\Pi}$ are finite as long as $\delta < \sqrt{\dfrac{2}{3\overline{\mu}^2}}$ or $\dfrac{d}{\lambda} < \dfrac{1}{7{\cdot}66 \times \overline{\mu}}$.

We shall, as before, neglect $\delta^4, \ldots$, but it will be necessary to go to order δ^3 in $\dfrac{dv}{d\xi}$ in order that the result should be correct to δ^2; we shall also neglect $\dfrac{1}{m^2}$ when multiplied by δ^2, since m^2 is about 100 ('B. A. Rep.,' 1885, p. 192).

We have

$$u^0 = u_0$$

$$(u')_{\xi=1} = - \tfrac{1}{2}\iota\nu\,(v_0 + v_1) + \iota\nu\,(v_1 - v_0)\int_0^1 \frac{\eta\,d\eta}{m^2} - \Pi_0\int_0^1 \frac{d\eta}{m^2}$$

$$(u'')_{\xi=1} = - \int_0^1 \frac{\Pi'}{m^2}\,d\eta = 0, \text{ } u'' \text{ being multiplied by } \delta^2$$

$$(\Pi^0)_{\xi=1} = \Pi_0 - \iota\nu\,(v_1 - v_0)$$

$$(\Pi')_{\xi=1} = u_0 \,.\, \left\{\int_0^1 \mu^2\,d\eta - \nu^2\right\}$$

$$(\Pi'')_{\xi=1} = \int_0^1 (\mu^2 - \nu^2)\, u'\, d\eta = -\,\iota\nu v_0 \int_0^1 (\mu^2 - \nu^2)\, \eta\,(1 - \tfrac{1}{2}\eta)\, d\eta - \tfrac{1}{2}\iota\nu v_1 \int_0^1 (\mu^2 - \nu^2)\, \eta^2\, d\eta$$

$$= \iota\nu v_0 \,.\left\{\tfrac{1}{3}\nu^2 - \int_0^1 \mu^2\eta\,(1 - \tfrac{1}{2}\eta)\, d\eta\right\} + \tfrac{1}{2}\iota\nu v_1 \left\{\tfrac{1}{3}\nu^2 - \int_0^1 \mu^2\eta^2\, d\eta\right\}$$

neglecting the terms in u' involving $\frac{1}{m^2}$.

$$\left(\frac{dv^0}{d\xi}\right)_{\xi=0} = v_1 - v_0, \quad \left(\frac{dv^0}{d\xi}\right)_{\xi=1} = v_1 - v_0$$

$$\left(\frac{dv'}{d\xi}\right)_{\xi=0} = 0, \quad \left(\frac{dv'}{d\xi}\right)_{\xi=1} = 0$$

$$\left(\frac{dv''}{d\xi}\right)_{\xi=0} = -\,\iota\nu \,.\int_0^1 \left(1 - \frac{1}{m^2}\right) \Pi^0 .\,(1 - \eta)\, d\eta + \int_0^1 (\mu^2 - \nu^2)\, v^0 .\,(1 - \eta)\, d\eta$$

$$= -\,\iota\nu \int_0^1 \left(1 - \frac{1}{m^2}\right) \{\Pi_0 - \iota\nu\,(v_1 - v_0)\,\eta\}\,(1 - \eta)\, d\eta$$

$$+ \int_0^1 (\mu^2 - \nu^2)\,\{v_0\,(1 - \eta) + v_1\eta\}\,(1 - \eta)\, d\eta$$

$$= v_0 .\left\{\int_0^1 \mu^2\,(1 - \eta)^2 .\, d\eta - \tfrac{1}{6}\,\nu^2 - \nu^2 \int_0^1 \frac{\eta\,(1 - \eta)\, d\eta}{m^2}\right\}$$

$$+ v_1 \left\{\int_0^1 \mu^2 .\,\eta\,(1 - \eta)\, d\eta - \tfrac{1}{3}\,\nu^2 + \nu^2 \int_0^1 \frac{\eta\,(1 - \eta)\, d\eta}{m^2}\right\}$$

$$-\,\iota\nu\Pi_0 .\left\{\tfrac{1}{2} - \int_0^1 \frac{(1 - \eta)\, d\eta}{m^2}\right\}$$

$$\left(\frac{dv''}{d\xi}\right)_{\xi=1} = \iota\nu \,.\int_0^1 \left(1 - \frac{1}{m^2}\right) \Pi^0 .\,\eta d\eta - \int_0^1 (\mu^2 - \nu^2)\, v^0 .\,\eta d\eta$$

$$= \iota\nu \int_0^1 \left(1 - \frac{1}{m^2}\right) \{\Pi_0 - \iota\nu\,(v_1 - v_0)\,\eta\}\,\eta d\eta - \int_0^1 (\mu^2 - \nu^2)\,\{v_0\,(1 - \eta) + v_1\eta\}\,\eta d\eta$$

$$= -\,v_0 .\left\{\int_0^1 \mu^2 .\,\eta\,(1 - \eta)\, d\eta + \tfrac{1}{6}\,\nu^2 - \nu^2 \int_0^1 \frac{\eta^2\, d\eta}{m^2}\right\}$$

$$-\,v_1 \left\{\int_0^1 \mu^2 .\,\eta^2\, d\eta - \tfrac{2}{3}\,\nu^2 + \nu^2 \int_0^1 \frac{\eta^2\, d\eta}{m^2}\right\} + \iota\nu\Pi_0 .\left\{\tfrac{1}{2} - \int_0^1 \frac{\eta\, d\eta}{m^2}\right\}$$

$$\left(\frac{dv'''}{d\xi}\right)_{\xi=0} = -\,\iota\nu \int_0^1 \left(1 - \frac{1}{m^2}\right) \Pi^1 .\,(1 - \eta)\, d\eta + \int_0^1 (\mu^2 - \nu^2)\, v^1 .\,(1 - \eta)\, d\eta$$

$$= -\,\iota\nu u_0 \int_0^1 \left\{\int_0^\eta \mu^2\, d\xi - \nu^2\eta\right\}(1 - \eta)\, d\eta$$

$$= -\,\tfrac{1}{2}\,\iota\nu u_0 .\left\{\int_0^1 \mu^2 .\,(1 - \eta)^2\, d\eta - \tfrac{1}{3}\,\nu^2\right\}, \text{ neglecting } \frac{1}{m^2}.$$

$$\left(\frac{dv'''}{d\xi}\right)_{\xi=1} = \iota\nu u_0 \int_0^1 \left\{\int_0^\eta \mu^2\, d\xi - \nu^2\eta\right\} \eta\, d\eta = \tfrac{1}{2}\,\iota\nu u_0 \left\{\int_0^1 \mu^2 .\,(1 - \eta^2)\, d\eta - \tfrac{2}{3}\,\nu^2\right\}.$$

Thus, we have finally

$$\left.\begin{aligned}
(u)_{\xi=1} &= u_0 - \tfrac{1}{2}\,\iota\nu\,\delta\,(v_0 + v_1) + \iota\nu\,\delta\,(v_1 - v_0)\int_0^1 \frac{\eta\,d\eta}{m^2} - \delta\Pi_0\int_0^1 \frac{d\eta}{m^2} \\
(\Pi)_{\xi=1} &= \Pi_0 + \iota\nu v_0.\left[1 - \delta^2.\left\{\int_0^1 \mu^2\eta\,(1 - \tfrac{1}{2}\,\eta)\,d\eta - \tfrac{1}{3}\,\nu^2\right\}\right] \\
&\quad - \iota\nu v_1\left[1 + \tfrac{1}{2}\,\delta^2\left\{\int_0^1 \mu^2\eta^2\,d\eta - \tfrac{1}{3}\,\nu^2\right\}\right] + \delta u_0\left\{\int_0^1 \mu^2\,d\eta - \nu^2\right\} \\
\left(\frac{dv}{d\xi}\right)_{\xi=} &= - v_0.\left[1 - \delta^2.\left\{\int_0^1 \mu^2\,(1 - \eta)^2\,d\eta - \tfrac{1}{6}\,\nu^2\right\}\right] \\
&\quad + v_1\left[1 + \delta^2.\left\{\int_0^1 \mu^2\eta\,(1 - \eta)\,d\eta - \tfrac{1}{3}\,\nu^2\right\}\right] - \tfrac{1}{2}\,\iota\nu\Pi_0.\delta^2 \\
\frac{1}{\delta}\left\{\left(\frac{dv}{d\xi}\right)_{\xi=1} - \left(\frac{dv}{d\xi}\right)_{\xi=0}\right\} &= - \delta v_0.\left[\int_0^1 \mu^2\,(1 - \eta)\,d\eta - \nu^2\int_0^1 \frac{\eta\,d\eta}{m^2}\right] \\
&\quad - \delta v_1.\left[\int_0^1 \mu^2\eta\,d\eta - \nu^2 + \nu^2\int_0^1 \frac{\eta\,d\eta}{m^2}\right] + \iota\nu\,\delta\Pi_0\left[1 - \int_0^1 \frac{d\eta}{m^2}\right] \\
&\quad + .\iota\nu\,\delta^2 u_0.\left[\int_0^1 \mu^2.(1 - \eta)\,d\eta - \tfrac{1}{2}\,\nu^2\right]
\end{aligned}\right\}\quad . \text{(VII.)}$$

§ 4. *Summary.*

The conditions determining r, r', . . . are

Vibrations perpendicular to plane of incidence, when

$$x = 0, \quad \frac{dw}{dx} = \iota\,\frac{2\pi\mu_0}{\lambda}\cos i_0\,(1 - r), \qquad \text{when } x = d, \quad \frac{dw}{dx} = \iota\,\frac{2\pi\mu_1}{\lambda}\cos i_1.s.$$

Vibrations parallel to plane of incidence

Electromagnetic Theory, when

$$x = 0, \quad \frac{dv}{dx} = \iota\,\frac{2\pi\mu_0}{\lambda}.\cos^2 i_0\,(1 - r), \qquad \text{when } x = d, \quad \frac{dv}{dx} = \iota\,\frac{2\pi\mu_1}{\lambda}\cos^2 i_1.s.$$

Elastic Solid Theory, when

$$x = 0, \quad \frac{dv}{dx} = \iota\,\frac{2\pi\mu_0}{\lambda}\{\cos^2 i_0\,(1 - r) + \alpha_0 \sin i_0.r'\}\ ;$$

when

$$x = d, \quad \frac{dv}{dx} = \iota\,\frac{2\pi\mu}{\lambda}\{\cos^2 i_1.s - \alpha_1 \sin i_1 s'\}, \; u = -(\sin i_1 s + \alpha_1 s'), \; \Pi_1 = \mu_1 s'.$$

Write

$$A = \frac{1}{d}\int_0^d \mu^2\,dx, \qquad B = \frac{1}{d^2}\int_0^d \mu^2 x^2\,dx, \qquad C = \frac{1}{d^2}\int_0^d \mu^2 (d-x)\,dx,$$

$$D = \frac{1}{d^3}\int_0^d \mu^2 x^2\,dx, \qquad E = \frac{1}{d^3}\int_0^d \mu^2 x(d-x)\,dx, \qquad F = \frac{1}{d^3}\int_0^d \mu^2 (d-x)^2\,dx,$$

$$\pi(x) = \frac{1}{d}\int_0^x \frac{\mu^2-\nu^2}{\mu^2}\,dx = \frac{x}{d} - \frac{\nu^2}{d}\int_0^x \frac{dx}{\mu^2}, \qquad G = \frac{1}{d}\int_0^d \frac{dx}{\mu^2},$$

$$J = \frac{1}{d^2}\int_0^d\!\!\int_0^x \left(\frac{{\mu_x}^2}{{\mu_\xi}^2} - \frac{{\mu_\xi}^2}{{\mu_x}^2}\right) d\xi\,dx, \qquad H = \frac{1}{d}\int_0^d \mu^2\pi(x)\,dx,$$

$$I = \frac{1}{d}\int_0^d \mu^2\{\pi(d)-\pi(x)\}\,dx, \qquad K = \frac{1}{d}\int_0^d \mu^2\{\pi(x)\}^2\,dx,$$

$$L = \frac{1}{d}\int_0^d \mu^2\pi(x)\{\pi(d)-\pi(x)\}\,dx, \qquad M = \frac{1}{d}\int_0^d \mu^2\{\pi(d)-\pi(x)\}^2\,dx,$$

$$A' = \frac{1}{d}\int_0^d \frac{dx}{m^2}, \qquad B' = \frac{1}{d^2}\int_0^d \frac{x\,dx}{m^2}, \qquad C' = \frac{1}{d^2}\int_0^d \frac{(d-x)\,dx}{m^2}.$$

Then there are the following relations between these constants—

$$D+E=B, \quad E+F=C, \quad D+2E+F=B+C=A, \quad D-F=B-C.$$

$$\pi(d) = 1-\nu^2 G = \alpha, \quad K+L = (1-\nu^2 G)\,H, \quad L+M = (1-\nu^2 G)\,I,$$

$$K+2L+M = (1-\nu^2 G)(H+I) = (1-\nu^2 G)^2.A = \alpha^2 A.$$

$$H = \frac{1}{d^2}\int_0^d \mu^2 \int_0^x \left(1-\frac{\nu^2}{\mu^2}\right) d\xi = B - \frac{\nu^2}{d^2}\int_0^d\!\!\int_0^x \frac{{\mu_x}^2}{{\mu_\xi}^2}\,.\,d\xi\,dx\,;$$

$$I = (1-\nu^2 G)\,A - H = C - \frac{\nu^2}{d^2}\int_0^d\!\!\int_0^x \frac{{\mu_\xi}^2}{{\mu_x}^2}\,.\,d\xi\,dx.$$

Hence

$$K - M = (1-\nu^2 G)(H-I) = (1-\nu^2 G)(B - C - \nu^2 J),$$

so that for $\nu = 0$,

$$K - M = B - C.$$

And lastly,

$$B' + C' = A'.$$

Using this notation we have for *Vibrations Perpendicular to Plane of Incidence*—

$$d\left(\frac{dw}{dx}\right)_{x=0} = -\,w_0\{1-\delta^2(F-\tfrac{1}{3}\nu^2)\} + w_1\{1+\delta^2(E-\tfrac{1}{6}\nu^2)\},$$

$$d\left(\frac{dw}{dx}\right)_{x=d} = -\,w_0\{1+\delta^2(E-\tfrac{1}{6}\nu^2)\} + w_1\{1-\delta^2(D-\tfrac{1}{3}\nu^2)\}\;.\;.\;.\quad (\text{V}'.)$$

Electromagnetic Theory. Vibrations Parallel to Plane of Incidence.

$$\begin{aligned} d\left(\frac{dv}{dx}\right)_{x=0} &= -\frac{\cos^2 i_0}{a} v_0 . \left(1 - \delta^2 \frac{\mathrm{M}}{a}\right) + \frac{\cos^2 i_0}{a} v_1 \left(1 + \delta^2 \frac{\mathrm{L}}{a}\right), \\ d\left(\frac{dv}{dx}\right)_{x=d} &= -\frac{\cos^2 i_1}{a} v_0 \left(1 + \delta^2 \frac{\mathrm{L}}{a}\right) + \frac{\cos^2 i_1}{a} v_1 . \left(1 - \delta^2 \frac{\mathrm{K}}{a}\right) \ldots \quad (\mathrm{VI}'.). \end{aligned}$$

Elastic Solid Theory—Vibrations Parallel to the Plane of Incidence.

$$\left.\begin{aligned} d\left(\frac{dv}{dx}\right)_{x=0} &= -v_0 . \{1 - \delta^2 . (\mathrm{F} - \tfrac{1}{6}\nu^2)\} \\ &\quad + v_1 . \{1 + \delta^2 . (\mathrm{E} - \tfrac{1}{6} v^2)\} - \tfrac{1}{2}\iota\nu\Pi_0 . \delta^2 \\ u_{x=d} &= u_0 - \tfrac{1}{2}\iota\nu\,\delta\,(v_1 + v_0) + \iota\nu\,\delta\,(v_1 - v_0)\,\mathrm{B}' - \delta\Pi_0\mathrm{A}' \\ \Pi_{x=d} &= \Pi_0 + \iota\nu v_0 . \{1 - \delta^2 . (\mathrm{B} - \tfrac{1}{2}\mathrm{D} - \tfrac{1}{3}\nu^2)\} \\ &\quad - \iota\nu v_1 . \{1 + \tfrac{1}{2}\delta^2 . (\mathrm{D} - \tfrac{1}{3}\nu^2)\} + \delta u_0 . (\mathrm{A} - \nu^2) \\ \frac{\lambda}{2\pi}\left\{\left(\frac{dv}{dx}\right)_{x=d} - \left(\frac{dv}{dx}\right)_{x=0}\right\} &= -\delta v_0 . (\mathrm{C} - \nu^2\mathrm{B}') - \delta v_1 . (\mathrm{B} - \nu^2 + \nu^2\mathrm{B}') \\ &\quad + \iota\nu\,\delta\Pi_0 . (1 - \mathrm{A}') + \iota\nu\,\delta^2 . u_0 . (\mathrm{C} - \tfrac{1}{2}\nu^2) \end{aligned}\right\} . (\mathrm{VII}'.).$$

where

$$u_0 = -\sin i_0\,(1 - r) + \alpha_0 r', \qquad v_0 = \cos i_0\,(1 + r) + \iota \sin i_0 . r', \qquad \Pi_0 = \mu_0 r',$$

$$w_0 = 1 + r, \qquad v_1 = \cos i_1 . s + \iota \sin i_1 . s', \qquad w_1 = s.$$

We found also that the series (V′) converge at least as rapidly as the geometrical progression $1 + \left(\frac{d}{\lambda} \times 4{\cdot}53 \times \bar{\mu}\right)^2 + \left(\frac{d}{\lambda} \times 4{\cdot}53 \times \bar{\mu}\right)^4 + \ldots$

The series (VI′.) converge at least as fast as

$$1 + \left(\frac{d}{\lambda} \times 6{\cdot}28 \times \bar{\mu}\right)^2 + \left(\frac{d}{\lambda} \times 6{\cdot}28 \times \bar{\mu}\right)^4 + \ldots$$

and the series (VII′.) at least as fast as

$$1 + \left(\frac{d}{\lambda} \times 7{\cdot}66 \times \bar{\mu}\right) + \left(\frac{d}{\lambda} \times 7{\cdot}66 \times \bar{\mu}\right)^2 + \ldots$$

where $\bar{\mu}$ denotes the greatest value μ has in the variable layer.

The greatest refractive index for transparent substances (excluding metals) occurs in Greenockite, and has the value 2·66. Taking this value for $\bar{\mu}$ the three ratios are

$12\frac{d}{\lambda}$, $17\frac{d}{\lambda}$, and $20\frac{d}{\lambda}$. If for $\bar{\mu}$ we take 1·5 they are $7\frac{d}{\lambda}$, $9\frac{d}{\lambda}$, $11\frac{d}{\lambda}$; if we take $\bar{\mu} = 1{\cdot}334$, the value for water, the ratios are $6\frac{d}{\lambda}$, $8\frac{d}{\lambda}$, $10\frac{d}{\lambda}$.

Reinold and Rücker found for the thickness of a black soap-film $\cdot 7 \times 10^{-6}$ to $1{\cdot}4 \times 10^{-6}$ cms., for red of the first order $2{\cdot}8 \times 10^{-5}$, blue of second order $3{\cdot}5 \times 10^{-5}$; these are in wave-lengths of yellow light $\frac{\lambda}{85} - \frac{\lambda}{43}$, $\frac{\lambda}{2}$, $\frac{3\lambda}{5}$. It follows that for the series to converge at all, the thickness of the film must be less than that of a soap-film giving the red of the first order.

§ 5. *Equations Determining the Constants r, r', . . .*

Vibrations Perpendicular to the Plane of Incidence.

The equations (V′.) (p. 837) give, on substituting for w_0 . . .

$$\begin{aligned}
&-(1+r)\{1-\delta^2(\mathrm{F}-\tfrac{1}{3}\nu^2)\}+s\{1+\delta^2(\mathrm{E}-\tfrac{1}{6}\nu^2)\}=\iota\,\delta\mu_0\cos i_0\,(1-r)\\
&-(1+r)\{1+\delta^2(\mathrm{E}-\tfrac{1}{6}\nu^2)\}+s\{1-\delta^2.(\mathrm{D}-\tfrac{1}{3}\nu^2)\}=\iota\,\delta\mu_1\cos i_1\,.\,s,
\end{aligned}$$

or

$$\begin{aligned}
&r.\{1-\iota\,\delta\mu_0\cos i_0-\delta^2.(\mathrm{F}-\tfrac{1}{3}\nu^2)\}-s.\{1+\delta^2(\mathrm{E}-\tfrac{1}{6}\nu^2)\}\\
&\quad=-\{1+\iota\,\delta\mu_0\cos i_0-\delta^2.(\mathrm{F}-\tfrac{1}{3}\nu^2)\},\\
&-r.\{1+\delta^2(\mathrm{E}-\tfrac{1}{6}\nu^2)\}+s.\{1-\iota\,\delta\mu_1\cos i_1-\delta^2(\mathrm{D}-\tfrac{1}{3}\nu^2)\}=1+\delta^2.(\mathrm{E}-\tfrac{1}{6}\nu^2),
\end{aligned}$$

whence

$$\begin{aligned}
r.[&\{1-\iota\,\delta\mu_0\cos i_0-\delta^2(\mathrm{F}-\tfrac{1}{3}\nu^2)\}\{1-\iota\,\delta\mu_1\cos i_1-\delta^2(\mathrm{D}-\tfrac{1}{3}\nu^2)\}\\
&-\{1+\delta^2(\mathrm{E}-\tfrac{1}{6}\nu^2)\}^2]\\
=-[&\{1+\iota\,\delta\mu_0\cos i_0-\delta^2(\mathrm{F}-\tfrac{1}{3}\nu^2)\}\left\{1-\iota\,\delta\mu_1\cos i_1-\delta^2\left(\mathrm{D}-\frac{\nu^2}{3}\right)\right\}\\
&-\{1+\delta^2(\mathrm{E}-\tfrac{1}{6}\nu^2)\}^2],
\end{aligned}$$

or

$$\begin{aligned}
&r\,[\mu_0\cos i_0+\mu_1\cos i_1-\iota\,\delta\,(\mathrm{A}+\mu_0\mu_1\cos i_0\cos i_1-\nu^2)\\
&\qquad-\delta^2.\{\mu_0\cos i_0(\mathrm{D}-\tfrac{1}{3}\nu^2)+\mu_1\cos i_1(\mathrm{F}-\tfrac{1}{3}\nu^2)\}],\\
&\quad=+[\mu_0\cos i_0-\mu_1\cos i_1+\iota\,\delta.(\mathrm{A}-\mu_0\mu_1\cos i_0\cos i_1-\nu^2)\\
&\qquad-\delta^2.\{\mu_0\cos i_0(\mathrm{D}-\tfrac{1}{3}\nu^2)-\mu_1\cos i_1(\mathrm{F}-\tfrac{1}{3}\nu^2)\}],\ \text{since } \mathrm{D}+2\mathrm{E}+\mathrm{F}\equiv\mathrm{A}\ \text{(p. 836)}.
\end{aligned}$$

Similarly

$$\begin{aligned}
&s.[\mu_0\cos i_0+\mu_1\cos i_1-i\,\delta\,(\mathrm{A}+\mu_0\mu_1\cos i_0\cos i_1-\nu^2)\\
&\quad-\delta^2.\{\mu_0\cos i_0(\mathrm{D}-\tfrac{1}{3}\nu^2)+\mu_1\cos i_1(\mathrm{F}-\tfrac{1}{3}\nu^2)]=2\mu_0\cos i_0\,.\,\{1+\delta^2.(\mathrm{E}-\tfrac{1}{6}\nu^2)\}.
\end{aligned}$$

Since $r = \mathrm{R}\epsilon^{\iota\rho}$, $s = \mathrm{S}\epsilon^{\iota\sigma}$, we get, by changing ι into $-\iota$, multiplying and dividing

$$\begin{aligned}
\mathrm{R}^2.[&(\mu_0\cos i_0+\mu_1\cos i_1)^2-2\,\delta^2.(\mu_0\cos i_0+\mu_1\cos i_1)\{\mu_0\cos i_0(\mathrm{D}-\tfrac{1}{3}\nu^2)\\
&+\mu_1\cos i_1(\mathrm{F}-\tfrac{1}{3}\nu^2)\}+\delta^2(\mathrm{A}+\mu_0\mu_1\cos \mathrm{i}_0\cos i_1-\nu^2)^2]\\
=\ &(\mu_0\cos i_0-\mu_1\cos i_1)^2-2\,\delta^2.(\mu_0\cos i_0-\mu_1\cos i_1)\{\mu_0\cos i_0(\mathrm{D}-\tfrac{1}{3}\nu^2)\\
&-\mu_1\cos i_1(\mathrm{F}-\tfrac{1}{3}\nu^2)\}+\delta^2.(\mathrm{A}-\mu_0\mu_1\cos i_0\cos i_1-\nu^2)^2
\end{aligned}$$

and

$$\mathrm{S}^2 . [(\mu_0 \cos i_0 + \mu_1 \cos i_1)^2 - 2\,\delta^2 (\mu_0 \cos i_0 + \mu_1 \cos i_1) \{\mu_0 \cos i_0 (\mathrm{D} - \tfrac{1}{3}\nu^2) + \mu_1 \cos i_1 (\mathrm{F} - \tfrac{1}{3}\nu^2)\} + \delta^2 (\mathrm{A} + \mu_0\mu_1 \cos i_0 \cos i_1 - \nu^2)^2]$$
$$= 4\mu_0^2 \cos^2 i_0 [1 + 2\,\delta^2 (\mathrm{E} - \tfrac{1}{6}\nu^2)]$$

and

$$\epsilon^{2\iota\rho} = \frac{\mu_0 \cos i_0 + \mu_1 \cos i_1 + \iota\delta (\mathrm{A} + \mu_0\mu_1 \cos i_0 \cos i_1 - \nu^2) - \delta^2 . \{\mu_0 \cos i_0 (\mathrm{D} - \frac{1}{3}\nu^2) + \mu_1 \cos i_1 (\mathrm{F} - \frac{1}{3}\nu^2)\}}{\mu_0 \cos i_0 + \mu_1 \cos i_1 - \iota\delta (\mathrm{A} + \mu_0\mu_1 \cos i_0 \cos i_1 - \nu^2) - \delta^2 . \{\mu_0 \cos i_0 (\mathrm{D} - \frac{1}{3}\nu^2) + \mu_1 \cos i_1 (\mathrm{F} - \frac{1}{3}\nu^2)\}} . \times$$
$$\frac{\mu_0 \cos i_0 - \mu_1 \cos i_1 + \iota\delta (\mathrm{A} - \mu_0\mu_1 \cos i_0 \cos i_1 - \nu^2) - \delta^2 \{\mu_0 \cos i_0 (\mathrm{D} - \frac{1}{3}\nu^2) - \mu_1 \cos i_1 (\mathrm{F} - \frac{1}{3}\nu^2)\}}{\mu_0 \cos i_0 - \mu_1 \cos i_1 - \iota\delta (\mathrm{A} - \mu_0\mu_1 \cos i_0 \cos i_1 - \nu^2) - \delta^2 \{\mu_0 \cos i_0 (\mathrm{D} - \frac{1}{3}\nu^2) - \mu_1 \cos i_1 (\mathrm{F} - \frac{1}{3}\nu^2)\}}$$

and

$$\epsilon^{2\iota\sigma} = \frac{\mu_0 \cos i_0 + \mu_1 \cos i_1 + \iota\delta (\mathrm{A} + \mu_0\mu_1 \cos i_0 \cos i_1 - \nu^2) - \delta^2 \{\mu_0 \cos i_0 (\mathrm{D} - \frac{1}{3}\nu^2) + \mu_1 \cos i_1 (\mathrm{F} - \frac{1}{3}\nu^2)\}}{\mu_0 \cos i_0 + \mu_1 \cos i_1 - \iota\delta (\mathrm{A} + \mu_0\mu_1 \cos i_0 \cos i_1 - \nu^2) - \delta^2 \{\mu_0 \cos i_0 (\mathrm{D} - \frac{1}{3}\nu^2) + \mu_1 \cos i_1 (\mathrm{F} - \frac{1}{3}\nu^2)\}} .$$

Hence

$$\mathrm{R}^2 = \left(\frac{-\mu_0 \cos i_0 + \mu_1 \cos i_1}{\mu_0 \cos i_0 + \mu_1 \cos i_1}\right)^2 . \Bigg[1 - 2\delta^2 \frac{(\mu_0 \cos i_0 + \mu_1 \cos i_1)\{\mu_0 \cos i_0 (\mathrm{D} - \frac{1}{3}\nu^2) - \mu_1 \cos i_1 (\mathrm{F} - \frac{1}{3}\nu^2)\} - (\mu_0 \cos i_0 - \mu_1 \cos i_1)\{\mu_0 \cos i_0 (\mathrm{D} - \frac{1}{3}\nu^2) + \mu_1 \cos i_1 (\mathrm{F} - \frac{1}{3}\nu^2)\}}{\mu_0^2 \cos^2 i_0 - \mu_1^2 \cos^2 i_1}$$
$$+ \delta^2 . \frac{(\mu_0 \cos i_0 + \mu_1 \cos i_1)^2 (\mathrm{A} - \mu_0\mu_1 \cos i_0 \cos i_1 - \nu^2)^2 - (\mu_0 \cos i_0 - \mu_1 \cos i_1)^2 . (\mathrm{A} + \mu_0\mu_1 \cos i_0 \cos i_1 - \nu^2)^2}{(\mu_0^2 \cos^2 i_0 - \mu_1^2 \cos^2 i_1)^2}\Bigg]$$

or

$$\mathrm{R}^2 = \frac{\sin^2 (i_0 - i_1)}{\sin^2 (i_0 + i_1)} \left[1 + 4\delta^2 . \mu_0\mu_1 \cos i_0 \cos i_1 . \frac{(\mathrm{A} - \mu_0^2)(\mathrm{A} - \mu_1^2) + (\mathrm{B} - \mathrm{C})(\mu_1^2 - \mu_0^2)}{(\mu_1^2 - \mu_0^2)^2}\right],$$

using the equations $\nu^2 = \mu_0^2 \sin^2 i_0 = \mu_1^2 \sin^2 i_1$ and $\mathrm{D} + \mathrm{E} = \mathrm{B}$, $\mathrm{E} + \mathrm{F} = \mathrm{C}$ (p. 837)

(VIII.).

$$\mathrm{S}^2 = \left(\frac{2\mu_0 \cos i_0}{\mu_0 \cos i_0 + \mu_1 \cos i_1}\right)^2 . \left[1 + 2\delta^2 (\mathrm{E} - \tfrac{1}{6}\nu^2) + 2\delta^2 . \frac{\mu_0 \cos i_0 (\mathrm{D} - \frac{1}{3}\nu^2) + \mu_1 \cos i_1 (\mathrm{F} - \frac{1}{3}\nu^2)}{\mu_0 \cos i_0 + \mu_1 \cos i_1} - \delta^2 \frac{(\mathrm{A} + \mu_0\mu_1 \cos i_0 \cos i_1 - \nu^2)^2}{(\mu_0 \cos i_0 + \mu_1 \cos i_1)^2}\right]$$

or

$$\mathrm{S}^2 = \frac{4 \sin^2 i_1 \cos^2 i_0}{\sin^2 (i_0 + i_1)} \left[1 - \delta^2 \frac{(\mathrm{A} - \mu_0^2)(\mathrm{A} - \mu_1^2) + (\mathrm{B} - \mathrm{C})(\mu_1^2 - \mu_0^2)}{(\mu_0 \cos i_0 + \mu_1 \cos i_1)^2}\right],$$

using the same conditions as for R^2,

$$\tan \rho = 2\delta\mu_0 \cos i_0 . \frac{\mu_1^2 - \mathrm{A}}{\mu_1^2 - \mu_0^2}, \text{ neglecting } \delta^3, \text{ \&c.}$$

$$\tan \sigma = 2\delta . \frac{\mathrm{A} + \mu_0\mu_1 \cos (i_0 + i_1)}{\mu_0 \cos i_0 + \mu\ \cos i}, \text{ neglecting } \delta^3.$$

These give at normal incidence

$$R^2 = \left(\frac{\mu_1 - \mu_0}{\mu_1 + \mu_0}\right)^2 . \left[1 + 4\,\delta^2 \mu_0 \mu_1 \frac{(A - \mu_0^2)(A - \mu_1^2) + (B - C)(\mu_1^2 - \mu_0^2)}{(\mu_1^2 - \mu_0^2)^2}\right],$$

$$S^2 = \frac{4\mu_0^2}{(\mu_1 + \mu_0)^2}\left[1 - \delta^2 \frac{(A - \mu_0^2)(A - \mu_1^2) + (B - C)(\mu_1^2 - \mu_0^2)}{(\mu_1 + \mu_0)^2}\right].$$

Vibrations Parallel to Plane of Incidence (Electromagnetic and Contractile Ether Theories).

The equations (VI′.) p. 838, give on substitution for v_0, v_1, since, in this case, r', s' are zero,

$$-\cos i_0 (1 + r)\left(1 - \delta^2 \frac{M}{a}\right) + \cos i_1 s\left(1 + \delta^2 \frac{L}{a}\right) = \iota\, \delta a \mu_0 (1 - r)$$

$$-\cos i_0 (1 + r)\left(1 + \delta^2 \frac{L}{a}\right) + \cos i_1 s\left(1 - \delta^2 \frac{K}{a}\right) = \iota\, \delta a \mu_1 s,$$

or,

$$\cos i_0 r . \left(1 - \iota\, \delta a \frac{\mu_0}{\cos i_0} - \delta^2 \frac{M}{a}\right) - \cos i_1 s\left(1 + \delta^2 \frac{L}{a}\right) = -\cos i_0 \left(1 + \iota\, \delta a \frac{\mu_0}{\cos i_0} - \delta^2 \frac{M}{a}\right)$$

$$-\cos i_0 r . \left(1 + \delta^2 \frac{L}{a}\right) + \cos i_1 s\left(1 - \iota\, \delta a \frac{\mu_1}{\cos i_1} - \delta^2 \frac{K}{a}\right) = \cos i_0 \left(1 + \delta^2 \frac{L}{a}\right),$$

whence

$$r\left[\left(1 - \iota\, \delta a \frac{\mu_0}{\cos i_0} - \delta^2 \frac{M}{a}\right)\left(1 - \iota\, \delta a \frac{\mu_1}{\cos i_1} - \delta^2 \frac{K}{a}\right) - \left(1 + \delta^2 \frac{L}{a}\right)^2\right]$$

$$= -\left(1 + \iota\, \delta a \frac{\mu_0}{\cos i_0} - \delta^2 \frac{M}{a}\right)\left(1 - \iota\, \delta a \frac{\mu_1}{\cos i_1} - \delta^2 \frac{K}{a}\right) + \left(1 + \delta^2 \frac{L}{a}\right)^2$$

$$s\left[\left(1 - \iota\, \delta a \frac{\mu_0}{\cos i_0} - \delta^2 \frac{M}{a}\right)\left(1 - \iota\, \delta a \frac{\mu_1}{\cos i_1} - \delta^2 \frac{K}{a}\right) - \left(1 + \delta^2 \frac{L}{a}\right)^2\right]$$

$$= -\,2\iota\, \delta a \frac{\mu_0}{\cos i_1}\left(1 + \delta^2 \frac{L}{a}\right),$$

or,

$$r\,.\left[\frac{\mu_1}{\cos i_1}+\frac{\mu_0}{\cos i_0}-\iota\,\delta\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}+A\right)-\delta^2\left(\frac{\mu_0}{\cos i_0}\frac{K}{a}+\frac{\mu_1}{\cos i_1}\frac{M}{a}\right)\right]$$

$$=-\left[\frac{\mu_1}{\cos i_1}-\frac{\mu_0}{\cos i_0}+\iota\,\delta\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}-A\right)+\delta^2\left(\frac{\mu_0}{\cos i_0}\frac{K}{a}-\frac{\mu_1}{\cos i_1}\frac{M}{a}\right)\right]$$

and

$$s\left[\frac{\mu_1}{\cos i_1}+\frac{\mu_0}{\cos i_0}-\iota\,\delta\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}+A\right)-\delta^2\left(\frac{\mu_0}{\cos i_0}\,.\,\frac{K}{a}+\frac{\mu_1}{\cos i_1}\frac{M}{a}\right)\right]$$

$$=\frac{2\mu_0}{\cos i_1}\left(1+\delta^2\frac{L}{a}\right),\text{ since } K+2L+M\equiv a^2A\text{ (p. 837).}$$

Since $r=R\epsilon^{\iota\rho}$, $s=S\epsilon^{\iota\sigma}$, changing ι into $-\iota$, multiplying and dividing, we have

$$R^2\,.\left[\left(\frac{\mu_1}{\cos i_1}+\frac{\mu_0}{\cos i_0}\right)^2-2\,\delta^2\left(\frac{\mu_1}{\cos i_1}+\frac{\mu_0}{\cos i_0}\right)\left(\frac{\mu_0}{\cos i_0}\frac{K}{a}+\frac{\mu_1}{\cos i_1}\frac{M}{a}\right)+\delta^2\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}+A\right)^2\right]$$

$$=\left(\frac{\mu_1}{\cos i_1}-\frac{\mu_0}{\cos i_0}\right)^2+2\,\delta^2.\left(\frac{\mu_1}{\cos i_1}-\frac{\mu_0}{\cos i_0}\right)\left(\frac{\mu_0}{\cos i_0}\frac{K}{a}-\frac{\mu_1}{\cos i_1}\frac{M}{a}\right)+\delta^2\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}-A\right)^2$$

$$S^2\,.\left[\left(\frac{\mu_1}{\cos i_1}+\frac{\mu_0}{\cos i_0}\right)^2-2\,\delta^2\left(\frac{\mu_1}{\cos i_1}+\frac{\mu_0}{\cos i_0}\right)\left(\frac{\mu_0}{\cos i_0}\frac{K}{a}+\frac{\mu_1}{\cos i_1}\,.\,\frac{M}{a}\right)+\delta^2\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}+A\right)^2\right]$$

$$=\frac{4\mu_0^2}{\cos^2 i_1}\left(1+2\,\delta^2\frac{L}{a}\right)$$

$$\epsilon^{2\iota\rho}=\frac{\frac{\mu_1}{\cos i_1}+\frac{\mu_0}{\cos i_0}+\iota\,\delta\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}+A\right)}{\frac{\mu_1}{\cos i_1}+\frac{\mu_0}{\cos i_0}-\iota\,\delta\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}+A\right)}\;\frac{\frac{\mu_1}{\cos i_1}-\frac{\mu_0}{\cos i_0}+\iota\,\delta\left(a\frac{\mu_1\mu_0}{\cos i_1\cos_{10}}-A\right)}{\frac{\mu_1}{\cos i_1}-\frac{\mu_0}{\cos i_0}-\iota\,\delta\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}-A\right)}$$

$$=\frac{\frac{\mu_1^2}{\cos^2 i_1}-\frac{\mu_0^2}{\cos^2 i_0}+2\iota\,\delta\frac{\mu_0}{\cos i_0}\left(a\frac{\mu_1^2}{\cos^2 i_1}-A\right)}{\frac{\mu_1^2}{\cos^2 i_1}-\frac{\mu_0^2}{\cos^2 i_0}-2\iota\,\delta\frac{\mu_0}{\cos i_0}\left(a\frac{\mu_1^2}{\cos^2 i_1}-A\right)}$$

$$\epsilon^{2\iota\sigma}=\frac{\frac{\mu_1}{\cos i_1}+\frac{\mu_0}{\cos i_0}+\iota\,\delta\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}+A\right)}{\frac{\mu_1}{\cos i_1}+\frac{\mu_0}{\cos i_0}-\iota\,\delta\left(a\frac{\mu_1\mu_0}{\cos i_1\cos i_0}+A\right)}.$$

Hence

$$\left.\begin{aligned}
R^2 &= \left(\frac{\dfrac{\mu_1}{\cos i_1} - \dfrac{\mu_0}{\cos i_0}}{\dfrac{\mu_1}{\cos i_1} + \dfrac{\mu_0}{\cos i_0}}\right)^2 \left[1 + 2\delta^2 \frac{\dfrac{\mu_0}{\cos i_0}\dfrac{K}{a} + \dfrac{\mu_1}{\cos i_1}\dfrac{M}{a}}{\dfrac{\mu_1}{\cos i_1} + \dfrac{\mu_0}{\cos i_0}} + 2\delta^2 \frac{\dfrac{\mu_0}{\cos i_0}\dfrac{K}{a} - \dfrac{\mu_1}{\cos i_1}\dfrac{M}{a}}{\dfrac{\mu_1}{\cos i_1} - \dfrac{\mu_1}{\cos i_0}}\right. \\
&\qquad \left. + \delta^2 \frac{\left(a\dfrac{\mu_1\mu_0}{\cos i_1 \cos i_0} - A\right)^2}{\left(\dfrac{\mu_1}{\cos i_1} - \dfrac{\mu_0}{\cos i_0}\right)^2} - \delta^2 \frac{\left(a\dfrac{\mu_1\mu_0}{\cos i_1 \cos i_0} + A\right)^2}{\left(\dfrac{\mu_1}{\cos i_1} + \dfrac{\mu_0}{\cos i_0}\right)^2}\right] \\
&= \frac{\tan^2(i_0 - i_1)}{\tan^2(i_0 + i_1)}\left[1 + 4\delta^2\mu_1\mu_0 \cos i_1 \cos i_0 \frac{B - C - J\,\mu_0\mu_1 \sin i_0 \sin i_1}{\mu_1{}^2\cos^2 i_0 - \mu_0{}^2\cos^2 i_1}\right. \\
&\qquad \left. + 4\delta^2\mu_0\mu_1 \cos i_0 \cos i_1 \frac{\{A - \mu_0{}^2 - (A - G\mu_0{}^4)\sin^2 i_0\}\{A - \mu_1{}^2 - (A - G\mu_1{}^4)\sin^2 i_1\}}{(\mu_1{}^2\cos^2 i_0 - \mu_0{}^2\cos^2 i_1)^2}\right] \\
S^2 &= \left(\frac{\dfrac{2\mu_0}{\cos i_1}}{\dfrac{\mu_1}{\cos i_1} + \dfrac{\mu_0}{\cos i_0}}\right)^2 \left[1 + 2\,\delta^2\frac{L}{a} + 2\,\delta^2 \frac{\dfrac{\mu_0}{\cos i_0}\dfrac{K}{a} + \dfrac{\mu_1}{\cos i_1}\dfrac{M}{a}}{\dfrac{\mu_1}{\cos i_1} + \dfrac{\mu_0}{\cos i_0}}\right. \\
&\qquad \left. - \delta^2 \frac{\left(a\dfrac{\mu_1\mu_0}{\cos i_1 \cos i_0} + A\right)^2}{\left(\dfrac{\mu_1}{\cos i_1} + \dfrac{\mu_0}{\cos i_0}\right)^2}\right] \\
&= \frac{4\sin^2 i_1 \cos^2 i_0}{\sin^2(i_0 + i_1)\cos^2(i_0 - i_1)}\left[1 - \delta^2.(B - C - J\,\mu_0\mu_1 \sin i_0 \sin i_1).\frac{\mu_1 \cos i_0 - \mu_0 \cos i_1}{\mu_1 \cos i_0 + \mu_0 \cos i_1}\right. \\
&\qquad \left. - \delta^2 \frac{\{A - \mu_0{}^2 - (A - G\mu_0{}^4)\sin^2 i_0\}\{A - \mu_1{}^2 - (A - G\mu_1{}^4)\sin^2 i_1\}}{(\mu_1 \cos i_0 + \mu_0 \cos i_1)^2}\right] \\
\tan\rho &= \frac{2\delta\dfrac{\mu_0}{\cos i_0}\left(a\dfrac{\mu_1{}^2}{\cos^2 i_1} - A\right)}{\dfrac{\mu_1{}^2}{\cos^2 i_1} - \dfrac{\mu_0{}^2}{\cos^2 i_0}} \\
&= 2\delta\mu_0 \cos i_0 \frac{\mu_1{}^2 - A + (A - G\mu_1{}^4)\sin^2 i_1}{\mu_1{}^2\cos^2 i_0 - \mu_0{}^2\cos^2 i_1}, \\
&\qquad \text{since } a \equiv 1 - \nu^2 G \text{ (p. 837), where } \nu = \mu_0 \sin i_0 = \mu_1 \sin i_1. \\
\tan\sigma &= \frac{\delta.\left(a\dfrac{\mu_1\mu_0}{\cos i_1 \cos i_0} + A\right)}{\dfrac{\mu_1}{\cos i_1} + \dfrac{\mu_0}{\cos i_0}} \\
&= \frac{\delta(\mu_0\mu_1 + A\cos i_0 \cos i_1 - G\mu_0{}^2\mu_1{}^2 \sin i_0 \sin i_1)}{\mu_1 \cos i_0 + \mu_0 \cos i_1}.
\end{aligned}\right\} \text{(IX.)}$$

When $i_0 = 0$, we have

$$\mathrm{R}^2 = . \left(\frac{\mu_1 - \mu_0}{\mu_1 + \mu_0}\right)^2 . \left[1 + 4\delta^2 \mu_0\mu_1 \frac{(\mathrm{A} - \mu_0{}^2)(\mathrm{A} - \mu_1{}^2) + (\mathrm{B} - \mathrm{C})(\mu_1{}^2 - \mu_0{}^2)}{(\mu_1{}^2 - \mu_0{}^2)^2}\right],$$

$$\mathrm{S}^2 = \frac{4\mu_0{}^2}{(\mu_1 + \mu_0)^2}\left[1 - \delta^2 \frac{(\mathrm{A} - \mu_0{}^2)(\mathrm{A} - \mu_1{}^2) + (\mathrm{B} - \mathrm{C})(\mu_1{}^2 - \mu_0{}^2)}{(\mu_1 + \mu_0{}^2)}\right],$$

the same as for vibrations perpendicular to the plane of incidence.

When $i_0 + i_1 = \frac{1}{2}\pi$, we have

$$\mathrm{R}^2 = \tfrac{1}{4}\,\delta^2 . \frac{(\mathrm{A} + \mathrm{G}\,\mu_0{}^2\mu_1{}^2 - \mu_0{}^2 - \mu_1{}^2)^2}{\mu_0{}^2 + \mu_1{}^2}, \quad \mathrm{S}^2 = \frac{\mu_0{}^2}{\mu_1{}^2} . \left\{1 - \tfrac{1}{4}\,\delta^2 . \frac{(\mathrm{A} + \mathrm{G}\,\mu_0{}^2\mu_1{}^2 - \mu_0{}^2 - \mu_1{}^2)^2}{\mu_0{}^2 + \mu_1{}^2}\right\}$$

$\tan\rho = \pm \tan\frac{1}{2}\pi$ according as

$$\mu_1{}^2 + \mu_0{}^2 \gtrless \mathrm{A} + \mathrm{G}\,\mu_0{}^2\mu_1{}^2, \quad \tan\sigma = \delta\,\frac{\mathrm{A} + \mu_0{}^2 + \mu_1{}^2 - \mathrm{G}\mu_0{}^2\mu_1{}^2}{2\sqrt{\mu_0{}^2 + \mu_1{}^2}} .$$

Vibrations parallel to plane of incidence (Elastic Solid Theory).

The equations (VII′.) p. 838, give on substitution for v_0, v_1, . . .

$$\begin{aligned}
&- \{\cos i_0 (1 + r) + \iota \sin i_0 . r'\}\{1 - \delta^2 (\mathrm{F} - \tfrac{1}{6}\nu^2)\} \\
&\qquad + \{\cos i_1 . s + \iota \sin i_1 . s'\}\{1 + \delta^2 (\mathrm{E} - \tfrac{1}{3}\nu^2)\} - \tfrac{1}{2}\iota\nu\,\delta^2\mu_0 r' \\
&\qquad = \iota\,\delta\mu_0\{\cos^2 i_0 (1 - r) + \alpha_0 \sin i_0 . r'\}
\end{aligned}$$

$$\begin{aligned}
&- \sin i_0 (1 - r) + \alpha_0 r' - \iota\nu\,\delta . \{\cos i_0 (1 + r) + \iota \sin i_0 . r'\}(\tfrac{1}{2} + \mathrm{B}') \\
&\qquad - \iota\nu\,\delta . \{\cos i_1 s + \iota \sin i_1 s'\}(\tfrac{1}{2} - \mathrm{B}') - \delta\mu_0\mathrm{A}' . r' = -(\sin i_1 . s + \alpha_1 s')
\end{aligned}$$

$$\begin{aligned}
&\mu_0 r' + \iota\nu . \{\cos i_0 (1 + r) + \iota \sin i_0 . r'\}\{1 - \delta^2 (\mathrm{B} - \tfrac{1}{2}\mathrm{D} - \tfrac{1}{3}\nu^2)\} \\
&\qquad - \iota\nu\{\cos i_1 . s + \iota \sin i_1 . s'\}\{1 + \tfrac{1}{2}\delta^2 (\mathrm{D} - \tfrac{1}{3}\nu^2)\} \\
&\qquad + \delta\{- \sin i_0 (1 - r) + \alpha_0 r'\}(\mathrm{A} - \nu^2) = \mu_1 s'
\end{aligned}$$

$$\begin{aligned}
&- \delta\{\cos i_0 (1 + r) + \iota \sin i_0 r'\}(\mathrm{C} - \nu^2\mathrm{B}') - \delta . \{\cos i_1 s + \iota \sin i_1 s'\}(\mathrm{B} - \nu^2 + \nu^2\mathrm{B}') \\
&\qquad + \iota\nu\,\delta\mu_0 (1 - \mathrm{A}') . r' + \iota\nu\,\delta^2 . \{- \sin i_0 (1 - r) + \alpha_0 r'\}(\mathrm{C} - \tfrac{1}{2}\nu^2) \\
&\qquad = \iota\mu_1\{\cos^2 i_1 s - \alpha_1 \sin i_1 . s'\} - \iota\mu_0\{\cos^2 i_0 (1 - r) + \alpha_0 \sin i_0 . r^1\}.
\end{aligned}$$

where we have throughout neglected $\frac{1}{m^2}$ except in terms of orders δ^0, δ'.

We get, collecting the terms,

$$\begin{aligned}
&\cos i_0 \,.\, r \,.\, \{1 - \iota\,\delta\mu_0 \cos i_0 - \delta^2 (\mathrm{F} - \tfrac{1}{6}\nu^2)\} - \cos i_1 \,.\, s\, \{1 + \delta^2 (\mathrm{E} - \tfrac{1}{3}\nu^2)\} \\
&\quad + \iota \sin i_0 \,.\, r' \{1 + \delta\mu_0\alpha_0 + \delta^2 \,.\, (\tfrac{1}{2}\mu_0{}^2 - \mathrm{F} + \tfrac{1}{6}\nu^2)\} - \iota \sin i_1 \,.\, s' \,.\, \{1 + \delta^2 (\mathrm{E} - \tfrac{1}{3}\nu^2)\} \\
&= - \cos i_0 \,.\, \{1 + \iota\,\delta\mu_0 \cos i_0 - \delta^2 (\mathrm{F} - \tfrac{1}{6}\nu^2)\} \quad \ldots\ldots\ldots\ldots \quad (1),
\end{aligned}$$

$$\begin{aligned}
&\sin i_0 \,.\, r \,.\, \{1 - \iota\,\delta\mu_0 \cos i_0 \,.\, (\tfrac{1}{2} + \mathrm{B}')\} + \sin i_1 s \,.\, \{1 - \iota\,\delta\mu_1 \cos i_1 (\tfrac{1}{2} - \mathrm{B}')\} \\
&\quad + r' \,.\, \{\alpha_0 + \delta\mu_0 \,.\, (\sin^2 i_0 \,.\, \overline{\tfrac{1}{2} + \mathrm{B}'} - \mathrm{A}')\} + s' \,.\, \{\alpha_1 + \delta\mu_1 \sin^2 i_1 (\tfrac{1}{2} - \mathrm{B}')\} \\
&= \sin i_0 \,.\, \{1 + \iota\,\delta\mu_0 \cos i_0 (\tfrac{1}{2} + \mathrm{B}')\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),
\end{aligned}$$

$$\begin{aligned}
&\mu_0 \sin i_0 \cos i_0 \,.\, r \,.\, \left\{1 - \iota\,\delta \frac{\mathrm{A} - \nu^2}{\mu_0 \cos i_0} - \delta^2 (\mathrm{B} - \tfrac{1}{2}\mathrm{D} - \tfrac{1}{3}\nu^2)\right\} \\
&\quad - \mu_1 \sin i_1 \cos i_1 \,.\, s \,.\, \{1 + \tfrac{1}{2}\delta^2 (\mathrm{D} - \tfrac{1}{3}\nu^2)\} \\
&\quad - \iota\mu_0 \cos^2 i_0 \,.\, r' \,.\, \left\{1 + \delta\alpha_0 \frac{\mathrm{A} - \nu^2}{\mu_0 \cos^2 i_0} + \delta^2 \tan^2 i_0 \,.\, (\mathrm{B} - \tfrac{1}{2}\mathrm{D} - \tfrac{1}{3}\nu^2)\right\} \\
&\quad + \iota\mu_1 \cos^2 i_1 \,.\, s' \,.\, \{1 - \tfrac{1}{2}\delta^2 \tan^2 i_1 (\mathrm{D} - \tfrac{1}{3}\nu^2)\} \\
&= - \mu_0 \sin i_0 \cos i_0 \,.\, \left\{1 + \iota\,\delta \frac{\mathrm{A} - \nu^2}{\mu_0 \cos i_0} - \delta^2 (\mathrm{B} - \tfrac{1}{2}\mathrm{D} - \tfrac{1}{3}\nu^2)\right\} \quad \ldots\ldots \quad (3),
\end{aligned}$$

$$\begin{aligned}
&\mu_0 \cos^2 i_0 \,.\, r \,.\, \left\{1 - \iota\,\delta \frac{\mathrm{C} - \nu^2\mathrm{B}'}{\mu_0 \cos i_0} - \delta^2 \tan^2 i_0 (\mathrm{C} - \tfrac{1}{2}\nu^2)\right\} \\
&\quad + \mu_1 \cos^2 i_1 \,.\, s \,.\, \left\{1 - \iota\,\delta \frac{\mathrm{B} - \nu^2 + \nu^2\mathrm{B}'}{\mu_1 \cos i_1}\right\} \\
&\quad - \mu_0 \sin i_0 \,.\, r' \left\{\alpha_0 - \delta \frac{\mathrm{C} - \nu^2\mathrm{B}' - \mu_0{}^2 (1 - \mathrm{A}')}{\mu_0} + \delta^2\alpha_0 (\mathrm{C} - \tfrac{1}{2}\nu^2)\right\} \\
&\quad - \mu_1 \sin i_1 \,.\, s' \,.\, \left\{\alpha_1 - \delta \frac{\mathrm{B} - \nu^2 + \nu^2\mathrm{B}'}{\mu_1}\right\} \\
&= \mu_0 \cos^2 i_0 \,.\, \left\{1 + \iota\,\delta \frac{\mathrm{C} - \nu^2\mathrm{B}'}{\mu_0 \cos i_0} - \delta^2 \tan^2 i_0 (\mathrm{C} - \tfrac{1}{2}\nu^2)\right\} \,.\ldots\ldots\ldots \quad (4).
\end{aligned}$$

In working with these equations we shall throughout neglect δ^3, δ^4, $\delta^2\mathrm{A}'$, $\mathrm{A}'^2 \,.\, \delta$, &c., &c.

We may thus in terms of order δ^2, interchange α_0, $\sin i_0$ and α_1, $\sin i_1$.

Multiplying (1) by $\nu = \mu_0 \sin i_0 = \mu_1 \sin i_1$ and subtracting (3) we have

$$\begin{aligned}
&\iota\,\delta \sin i_0 \,.\, r\, \{\mathrm{A} - \mu_0{}^2 - \iota\,\delta\mu_0 \cos i_0 (\mathrm{B} - \tfrac{1}{2}\mathrm{D} - \mathrm{F} - \tfrac{1}{6}\nu^2)\} \\
&\quad - \delta^2 \,.\, \mu_1 \sin i_1 \cos i_1 \,.\, s \,.\, (\mathrm{E} - \tfrac{1}{2}\mathrm{D} - \tfrac{1}{6}\nu^2) \\
&\quad + \iota r' \,.\, \{\mu_0 + \delta\alpha_0\mathrm{A} + \mu_0\delta^2 \sin^2 i_0 (\mathrm{B} - \tfrac{1}{2}\mathrm{D} - \mathrm{F} + \tfrac{1}{2}\mu_0{}^2 - \tfrac{1}{6}\nu^2)\} \\
&\quad - \iota s' \,.\, \{\mu_1 + \mu_1 \delta^2 \sin^2 i_1 \,.\, (\mathrm{E} - \tfrac{1}{2}\mathrm{D} - \tfrac{1}{6}\nu^2)\} \\
&= \iota\,\delta \sin i_0 \,.\, \{\mathrm{A} - \mu_0{}^2 + \iota\,\delta\mu_0 \cos i_0 (\mathrm{B} - \tfrac{1}{2}\mathrm{D} - \mathrm{F} - \tfrac{1}{6}\nu^2)\} \quad \ldots \quad (5).
\end{aligned}$$

Similarly, from (2) and (4)

$$\begin{aligned} r\,.\,\{\mu_0 - \iota\delta\cos i_0\,(\mathrm{C} + \tfrac{1}{2}\nu^2) - \mu_0\,\delta^2\sin^2 i_0\,(\mathrm{C} - \tfrac{1}{2}\nu^2)\} + s\,.\,\{\mu_1 - \iota\,\delta\cos i_1\,(\mathrm{B} - \tfrac{1}{2}\nu^2)\} \\ + \delta r'\sin i_0\,.\,\{\mathrm{C} + \tfrac{1}{2}\nu^2 - \mu_0{}^2 - \delta\nu\,(\mathrm{C} - \tfrac{1}{2}\nu^2)\} + \delta s'\sin i_1\,.\,(\mathrm{B} - \tfrac{1}{2}\nu^2) \\ = \mu_0 + \iota\,\delta\cos i_0\,(\mathrm{C} + \tfrac{1}{2}\nu^2) - \mu_0\,\delta^2\sin^2 i_0\,.\,(\mathrm{C} - \tfrac{1}{2}\nu^2) \quad \ldots\ldots\ldots \quad (6). \end{aligned}$$

Multiply (1) by $\alpha_1\left\{1 - \delta\left(\tfrac{1}{2}\mu_1\alpha_1 + \nu\mathrm{B}' - \tfrac{1}{2}\mathrm{A}'\dfrac{\mu_0}{\sin i_0}\right) + \tfrac{1}{2}\,\delta^2\,.\,(\mathrm{F} - \mathrm{E} - \tfrac{1}{2}\mu_0{}^2 + \tfrac{2}{3}\nu^2)\right\}$,

(2) by $\iota\sin i_1\,.\left\{1 - \delta\left(\nu - \tfrac{1}{2}\mathrm{A}'\dfrac{\mu_0}{\sin i_0}\right) + \tfrac{1}{2}\,\delta^2\,.\,(\mathrm{C} - \tfrac{1}{2}\mu_0{}^2 + \nu^2)\right\}$,

and add, using the relations $\alpha_0 = \sqrt{\left(\sin^2 i_0 - \dfrac{1}{m_0{}^2}\right)}$, $\alpha_1 = \sqrt{\left(\sin^2 i_1 - \dfrac{1}{m_1{}^2}\right)}$,

whence $\qquad \dfrac{\alpha_0}{\sin i_0} + \dfrac{\sin i_0}{\alpha_0} = \dfrac{\alpha_1}{\sin i_1} + \dfrac{\sin i_1}{\alpha_1} = 2 + \text{terms in } \dfrac{1}{m^4}\ldots,$

and neglecting terms of order δ^3, $\delta^2\dfrac{1}{m^2}$, $\dfrac{1}{m^4}$, . . . we have

$$\left.\begin{aligned} &\iota r'\,.\,(\alpha_1\sin i_0 + \alpha_0\sin i_1) \\ &= -\,r\Big[(\alpha_1\cos i_0 + \iota\sin i_0\sin i_1)\,\{1 - \iota\,\delta\mu_0\epsilon^{-\iota i_0} - \tfrac{1}{2}\,\delta^2\,(\mathrm{C} + \tfrac{1}{2}\mu_0{}^2)\,\epsilon^{-2\iota i_0}\} \\ &\qquad\qquad + \tfrac{1}{2}\,\delta\mathrm{A}'\,.\,\frac{\mu_0{}^2}{\mu_1}\,.\,\epsilon^{\iota i_0}\Big] \\ &\quad + s\Big[(\alpha_1\cos i_1 - \iota\sin^2 i_1)\,\{1 - \delta\nu + \tfrac{1}{2}\delta^2\,.\,(\mathrm{C} - \tfrac{1}{2}\mu_0{}^2 + \nu^2)\} + \tfrac{1}{2}\delta\mathrm{A}'\,.\,\frac{\mu_0{}^2}{\mu_1}\,.\,\epsilon^{-\iota i_1}\Big] \\ &\quad - \Big[(\alpha_1\cos i_0 - \iota\sin i_0\sin i_1)\,\{1 + \iota\,\delta\mu_0\epsilon^{\iota i_0} - \tfrac{1}{2}\,\delta^2\,(\mathrm{C} + \tfrac{1}{2}\mu_0{}^2)\,\epsilon^{2\iota i_0}\} \\ &\qquad\qquad + \tfrac{1}{2}\,\delta\mathrm{A}'\,\frac{\mu_0{}^2}{\mu_1}\,.\,\epsilon^{-\iota i_0}\Big] \end{aligned}\right\}\ (7).$$

Again, multiplying

(1) by $\alpha_0\left\{1 - \delta\left(\tfrac{1}{2}\mu_0\alpha_0 - \nu\mathrm{B}' + \tfrac{1}{2}\mathrm{A}'\dfrac{\mu_0}{\sin i_0}\right) + \tfrac{1}{2}\delta^2\,.\,(\mathrm{F} - \mathrm{E} - \tfrac{1}{2}\mu_0{}^2 + \tfrac{2}{3}\nu^2)\right\}$,

(2) by $\iota\sin i_0\,.\left\{1 + \tfrac{1}{2}\,\delta\dfrac{\mu_0}{\sin i_0}\left(\mathrm{A}' - \dfrac{1}{m_0{}^2}\right) - \tfrac{1}{2}\,\delta^2\,.\,(\mathrm{C} - \tfrac{1}{2}\mu_0{}^2)\right\}$,

and subtracting, we find

$$\left.\begin{aligned}
&\iota s' (\alpha_1 \sin i_0 + \alpha_0 \sin i_1) \\
&\quad = r\Big[(\alpha_0 \cos i_0 - \iota \sin^2 i_0)\,\{1 - \iota\,\delta\mu_0 \cos i_0 - \tfrac{1}{2}\,\delta^2 (\mathrm{C} + \tfrac{1}{2}\mu_0^2 - \nu^2)\} \\
&\qquad\qquad - \tfrac{1}{2}\,\delta\left(\mathrm{A}' - \frac{1}{m_0^2}\right)\mu_0\,.\,\epsilon^{\iota i_0}\Big] \\
&\quad - s\Big[(\alpha_0 \cos i_1 + \iota \sin i_0 \sin i_1)\,\{1 + \tfrac{1}{2}\,\delta^2\,.\,(\mathrm{C} - \tfrac{1}{2}\mu_0^2)\,\epsilon^{-2\iota i_1}\} \\
&\qquad\qquad - \tfrac{1}{2}\,\delta\,.\left(\mathrm{A}' - \frac{1}{m_0^2}\right).\,\mu_0\epsilon^{-\iota i_1}\Big] \\
&\quad + \Big[(\alpha_0 \cos i_0 + \iota \sin^2 i_0)\,\{1 + \iota\,\delta\mu_0 \cos i_0 - \tfrac{1}{2}\,\delta^2\,.\,(\mathrm{C} + \tfrac{1}{2}\mu_0^2 - \nu^2)\} \\
&\qquad\qquad - \tfrac{1}{2}\,\delta\left(\mathrm{A}' - \frac{1}{m_0^2}\right).\,\mu^0\epsilon^{-\iota i_0}\Big]
\end{aligned}\right\}\quad (8).$$

Substituting from (7) and (8) for r', s' in (5) and rearranging the terms, we find

$$r.\left[\begin{aligned}
&\{\mu_0 (\alpha_1 \cos i_0 + \iota \sin i_0 \sin i_1) + \mu_1 (\alpha_0 \cos i_0 - \iota \sin^2 i_0)\}\,(1 + \tfrac{1}{2}\,\delta^2\nu^2) \\
&\quad + \delta.(1 - \iota\,\delta\mu_0 \cos i_0)\,\mathrm{A}\alpha_1 (\alpha_0 \cos i_0 - \iota \sin^2 i_0) - \iota\,\delta\mu_0^2\alpha_1 \cos 2i_0 \\
&\quad - \iota\,\delta\mu_0\mu_1\alpha_0 \cos i_0.\,\epsilon^{-\iota i_0} - \tfrac{1}{2}\,\delta\left(\mathrm{A}' - \frac{1}{m_0^2}\right)\frac{\mu_0}{\mu_1}\,.\,(\mu_1^2 - \mu_0^2)\,\epsilon^{\iota i_0} \\
&\quad + \tfrac{1}{2}\,\delta\mu_0\,\mu_1 \cos i_0 \frac{1}{m_1^2} - \tfrac{1}{2}\,\delta\,\frac{\mu_0}{\mu_1}\,(\mu_1^2 - \mu_0^2) \cos i_0 \frac{1}{m_0^2} - \delta^2\nu^2\,\mu_0 \sin i_1 \epsilon^{-} \\
&\quad - \tfrac{1}{2}\,\delta^2.(\mu_0 \sin i_1 \cos 2i_0 + \mu_1 \sin i_0 \cos 2i_1)\,(\mathrm{C} + \tfrac{1}{2}\,\mu_0^2)\,\epsilon^{-\iota i_0}
\end{aligned}\right]$$

$$-s.\left[\begin{aligned}
&\mu_0 (\alpha_1 \cos i_1 - \iota \sin^2 i_1) + \mu_1 (\alpha_0 \sin i_1 + \iota \sin i_0 \sin i_1) \\
&\quad + \delta\,(\mathrm{A} - \mu_0^2)\,\alpha_0 (\alpha_1 \cos i_1 - \iota \sin^2 i_1) - \tfrac{1}{2}\,\delta\left(\mathrm{A}' - \frac{1}{m_0^2}\right)\frac{\mu_0}{\mu_1}.(\mu_1^2 - \mu_0^2)\,\epsilon^{-\iota i_1} \\
&\quad + \tfrac{1}{2}\,\delta^2 (\mu_0 \sin i_1 \cos 2i_0 + \mu_1 \sin i_0 \cos 2i_1)\,(\mathrm{C} - \tfrac{1}{2}\,\mu_0^2)\,\epsilon^{-\iota i_1}
\end{aligned}\right]$$

$$= -\left[\begin{aligned}
&\{\mu_0 (\alpha_1 \cos i_0 - \iota \sin i_0 \sin i_1) + \mu_1 (\alpha_0 \cos i_0 + \iota \sin^2 i_0)\}\,(1 + \tfrac{1}{2}\,\delta^2\nu^2) \\
&\quad + \delta\,(1 + \iota\,\delta\mu_0 \cos i_0)\,\mathrm{A}\alpha_1 (\alpha_0 \cos i_0 + \iota \sin^2 i_0) + \iota\,\delta\mu_0^2\alpha_1 \cos 2i_0 \\
&\quad + \iota\,\delta\mu_0\mu_1\alpha_0 \cos i_0\epsilon^{\iota i_0} - \tfrac{1}{2}\,\delta\left(\mathrm{A}' - \frac{1}{m_0^2}\right)\frac{\mu_0}{\mu_1}.(\mu_1^2 - \mu_0^2)\,\epsilon^{-\iota i_0} \\
&\quad + \tfrac{1}{2}\,\delta\mu_0\mu_1 \cos i_0 \frac{1}{m_1^2} - \tfrac{1}{2}\,\delta\,\frac{\mu_0}{\mu_1}\,(\mu_1^2 - \mu_0^2) \cos i_0 \frac{1}{m_0^2} - \delta^2\nu^2\,\mu_0 \sin i_1.\,\epsilon^{\iota i_0} \\
&\quad - \tfrac{1}{2}\,\delta^2.(\mu_0 \sin i_1 \cos 2i_0 + \mu_1 \sin i_0 \cos 2i_1)\,(\mathrm{C} + \tfrac{1}{2}\mu_0^2)\,\epsilon^{\iota i_0}
\end{aligned}\right]$$

In the same way we get from (6)

$$r.\left[\begin{array}{l}\mu_0.(\alpha_1 \sin i_0 + \alpha_0 \sin i_1)(1 - \iota\,\delta\mu_0 \cos i_0) - \iota\,\delta \sin i_1 (\alpha_0 \cos i_0 - \iota \sin^2 i_0)(\mathrm{A} - \mu_0{}^2) \\ \qquad - \delta^2\mu_0 \cos i_0 \sin i_0 \sin i_1 \{\mu_0{}^2 \cos i_0 + (\mathrm{B} - \mathrm{C})\,\epsilon^{-\iota i_0}\}\end{array}\right]$$

$$+ s.\left[\begin{array}{l}\mu_1 (\alpha_1 \sin i_0 + \alpha_0 \sin i_1) - \iota\,\delta \sin i_0 (\alpha_1 \cos i_1 - \iota \sin^2 i_1)(\mathrm{A} - \mu_0{}^2) \\ \qquad + 2\,\iota\,\delta^2 \nu \sin i_0 \sin i_1 (\mathrm{C} - \tfrac{1}{2}\mu_0{}^2)\,\epsilon^{-\iota i_1}\end{array}\right]$$

$$\begin{array}{l}= \mu_0 (\alpha_1 \sin i_0 + \alpha_0 \sin i_1)(1 + \iota\,\delta\mu_0 \cos i_0) + \iota\delta \sin i_1.(\alpha_0 \cos i_0 + \iota \sin^2 i_0)(\mathrm{A} - \mu_0{}^2) \\ \qquad - \delta^2\mu_0 \cos i_0 \sin i_0 \sin i_1 \{\mu_0{}^2 \cos i_0 + (\mathrm{B} - \mathrm{C})\,\epsilon^{\iota i_0}\}.\end{array}$$

Solving these two equations for r, s, we get, after some algebraic transformations, using the values $\alpha_0 \sin i_0 = \sin^2 i_0 - \frac{1}{2m_0{}^2}$, $\alpha_1 \sin i_1 = \sin^2 i_1 - \frac{1}{2m_1{}^2}$, neglecting $\frac{1}{m^2}$ in terms of order δ^2, and finally discarding a common factor, $\alpha_0 \sin i_1 + \alpha_1 \sin i_0$—

$$r.\left[\begin{array}{l}\{1 - \iota\,\delta\mu_0 \cos i_0 - \tfrac{1}{2}\delta^2\mu_0{}^2 \cos^2 i_0 + \tfrac{1}{2}\delta^2 (\mathrm{C} - \tfrac{1}{2}\mu_0{}^2)\}\,\{\mu_0{}^2.(\alpha_1 \cos i_1 - \iota \sin^2 i_1) \\ \qquad\qquad + \mu_1{}^2.(\alpha_0 \cos i_0 - \iota \sin^2 i_0) + \mu_0\mu_1 (\alpha_1 \cos i_0 + \alpha_0 \cos i_1 + 2\iota \sin i_0 \sin i_1)\} \\ - \iota\,\delta.\{(1 - \iota\,\delta\mu_0 \cos i_0)(\mathrm{A} - \mu_0{}^2) - \iota\,\delta (\mu_1 \cos i_1 - \mu_0 \cos i_0)(\mathrm{C} - \tfrac{1}{2}\mu_0{}^2)\} \\ \qquad \{\mu_0 (\cos i_0 + \iota\alpha_0)(\alpha_1 \cos i_1 - \iota \sin^2 i_1) + \mu_1 (\cos i_1 + \iota\alpha_1)(\alpha_0 \cos i_0 - \iota \sin^2 i_0)\} \\ - \iota\,\delta^2.\{(\mathrm{A} - \mu_0{}^2)(\mathrm{A} - \mu_1{}^2) + (\mu_1{}^2 - \mu_0{}^2)(\mathrm{B} - \mathrm{C})\} \sin i_0 \sin i_1.\epsilon^{-\iota(i_0+i_1)} \\ - \tfrac{1}{2}\delta \left(\mathrm{A}' - \dfrac{1}{m_0{}^2}\right) \dfrac{\mu_0}{\mu_1} (\mu_1{}^2 - \mu_0{}^2)(\mu_0 \epsilon^{-\iota i_1} + \mu_1 \epsilon^{\iota i_0})\end{array}\right]$$

$$= - \left[\begin{array}{l}\{1 + \iota\,\delta\mu_0 \cos i_0 - \tfrac{1}{2}\delta^2\mu_0{}^2 \cos^2 i_0 + \tfrac{1}{2}\delta^2 (\mathrm{C} - \tfrac{1}{2}\mu_0{}^2)\}\,\{- \mu_0{}^2 (\alpha_1 \cos i_1 - i \sin^2 i_1) \\ \qquad\qquad + \mu_1{}^2.(\alpha_0 \cos i_0 + \iota \sin^2 i_0) - \mu_0\mu_1 (\alpha_0 \cos i_1 - \alpha_1 \cos i_0 + 2\iota \sin i_0 \sin i_1)\} \\ - \iota\,\delta.\{(1 + \iota\,\delta\mu_0 \cos i_0)(\mathrm{A} - \mu_0{}^2) - \iota\,\delta (\mu_1 \cos i_1 + \mu_0 \cos i_0)(\mathrm{C} - \tfrac{1}{2}\mu_0{}^2)\} \\ \qquad \{\mu_0 (\cos i_0 - \iota\alpha_0)(\alpha_1 \cos i_1 - \iota \sin^2 i_1) + \mu_1 (\cos i_1 + \iota\alpha_1)(\alpha_0 \cos i_0 + \iota \sin^2 i_0)\} \\ - \iota\,\delta^2.\{(\mathrm{A} - \mu_0{}^2)(\mathrm{A} - \mu_1{}^2) + (\mu_1{}^2 - \mu_0{}^2)(\mathrm{B} - \mathrm{C})\} \sin i_0 \sin i_1 \epsilon^{\iota(i_0-i_1)} \\ - \tfrac{1}{2}\delta \left(\mathrm{A}' - \dfrac{1}{m_0{}^2}\right) \dfrac{\mu_0}{\mu_1} (\mu_1{}^2 - \mu_0{}^2)(\mu_1 \epsilon^{-\iota i_0} - \mu_0 \epsilon^{-\iota i_1});\end{array}\right]$$

and, in the same way,

$$s.\left[\begin{array}{l} \{1 - \iota\,\delta\mu_0 \cos i_0 - \tfrac{1}{2}\,\delta^2\mu_0^2 \cos^2 i_0 + \tfrac{1}{2}\,\delta^2.(\mathrm{C} - \tfrac{1}{2}\,\mu_0^2)\}\ \{\mu_0^2.(\alpha_1 \cos i_1 - \iota \sin^2 i_1) \\ \qquad\qquad + \mu_1^2(\alpha_0 \cos i_0 - \iota \sin^2 i_0) + \mu_0\mu_1(\alpha_1 \cos i_0 + \alpha_0 \cos i_1 + 2\iota \sin i_0 \sin i_1)\} \\ - \iota\,\delta.\{(1 - \iota\,\delta\mu_0 \cos i_0)(\mathrm{A} - \mu_0^2) - \iota\,\delta(\mu_1 \cos i_1 - \mu_0 \cos i_0)(\mathrm{C} - \tfrac{1}{2}\,\mu_0^2)\} \\ \qquad \{\mu_0(\cos i_0 + \iota\alpha_0)(\alpha_1 \cos i_1 - \iota \sin^2 i_1) + \mu_1(\cos i_1 + \iota\alpha_1)(\alpha_0 \cos i_0 - \iota \sin^2 i_0)\} \\ - \iota\,\delta^2.\{(\mathrm{A} - \mu_0^2)(\mathrm{A} - \mu_1^2) + (\mu_1^2 - \mu_0^2)(\mathrm{B} - \mathrm{C})\} \sin i_0 \sin i_1 .\,\epsilon^{-\iota(i_0+i_1)} \\ - \tfrac{1}{2}\,\delta\left(\mathrm{A}' - \dfrac{1}{m_0^2}\right)\dfrac{\mu_0}{\mu_1}(\mu_1^2 - \mu_0^2)(\mu_0\epsilon^{-\iota i_1} + \mu_1\epsilon^{\iota i_0}) \end{array}\right]$$

$$= 2\mu_0 \cos i_0 .\left[\begin{array}{l}(\mu_0\alpha_1 + \mu_1\alpha_0)\{1 + \tfrac{1}{2}\,\delta^2(\mathrm{C} - \tfrac{1}{2}\,\mu_0^2)\} \\ \qquad - \delta.(\sin i_0 \sin i_1 - \alpha_0\alpha_1)(\mathrm{A} - \mu_0^2) - \tfrac{1}{2}\,\delta\left(\mathrm{A}' - \dfrac{1}{m_0^2}\right)\dfrac{\mu_0}{\mu_1}(\mu_1^2 - \mu_0^2)\end{array}\right].$$

We easily find

$$\mu_0^2(\alpha_1 \cos i_1 - \iota \sin^2 i_1) + \mu_1^2(\alpha_0 \cos i_0 - \iota \sin^2 i_0) + \mu_0\mu_1(\alpha_1 \cos i_0 + \alpha_0 \cos i_1 + 2\iota \sin i_0 \sin i_1)$$

$$= \frac{\mu_0\mu_1}{\nu} \sin(i_0 + i_1)\{(\mu_0\alpha_1 + \mu_1\alpha_0)\cos(i_0 - i_1) - \iota(\mu_1 \sin i_0 - \mu_0 \sin i_1)\sin(i_0 - i_1)\},$$

$$-\mu_0^2(\alpha_1 \cos i_1 - \iota \sin^2 i_1) + \mu_1^2(\alpha_0 \cos i_0 + \iota \sin^2 i_0) - \mu_0\mu_1(\alpha_0 \cos i_1 - \alpha_1 \cos i_0 + 2\iota \sin i_0 \sin i_1)$$

$$= \frac{\mu_0\mu_1}{\nu} \sin(i_0 - i_1)\{(\mu_0\alpha_1 + \mu_1\alpha_0)\cos(i_0 + i_1) + \iota(\mu_1 \sin i_0 - \mu_0 \sin i_1)\sin(i_0 + i_1)\},$$

$$\mu_0.\epsilon^{-\iota i_1} + \mu_1\epsilon^{\iota i_0} = \frac{\mu_0\mu_1}{\nu}\sin(i_0 + i_1).\epsilon^{\iota(i_0 - i_1)}, \quad \mu_1\epsilon^{-\iota i_0} - \mu_0\epsilon^{-\iota i_1} = \frac{\mu_0\mu_1}{\nu}\sin(i_0 - i_1).\epsilon^{-\iota(i_0+i_1)}.$$

$$\mu_0(\cos i_0 + \iota\alpha_0)(\alpha_1 \cos i_1 - \iota \sin^2 i_1) + \mu_1(\cos i_1 + \iota\alpha_1)(\alpha_0 \cos i_0 - \iota \sin^2 i_0)$$

$$= \cos(i_0 - i_1).(\mu_0\alpha_1 + \mu_1\alpha_1)\left\{1 - \iota\frac{\mu_0\mu_1 \sin(i_0 + i_1)}{(\mu_1^2 + \mu_0^2)\sin i_0 \sin i_1}(\sin i_0 \sin i_1 - \alpha_0\alpha_1)\right\}$$

$$- \iota \sin(i_0 - i_1)(\mu_1 \sin i_0 - \mu_0 \sin i_1)\left\{1 - \iota\frac{\mu_0\mu_1 \sin(i_0 + i_1)}{(\mu_1^2 - \mu_0^2)\sin i_0 \sin i_1}(\alpha_0 \sin i_1 - \alpha_1 \sin i_0)\right\}$$

$$\mu_0(\cos i_0 - \iota\alpha_0)(\alpha_1 \cos i_1 - \iota \sin^2 i_1) + \mu_1(\cos i_1 + \iota\alpha_1)(\alpha_0 \cos i_0 + \iota \sin^2 i_0)$$

$$= \cos(i_0 + i_1)(\mu_0\alpha_1 + \mu_1\alpha_0)\left\{1 - \iota\frac{\mu_0\mu_1 \sin(i_0 - i_1)}{(\mu_1^2 + \mu_0^2)\sin i_0 \sin i_1}(\sin i_0 \sin i_1 - \alpha_0\alpha_1)\right\}$$

$$+ \iota \sin(i_0 + i_1)(\mu_1 \sin i_0 - \mu_0 \sin i_1)\left\{1 - \iota\frac{\mu_0\mu_1 \sin(i_0 - i_1)}{(\mu_1^2 - \mu_0^2)\sin i_0 \sin i_1}(\alpha_0 \sin i_1 - \alpha_1 \sin i_0)\right\}.$$

Writing $\frac{\mu_1 \sin i_0 - \mu_0 \sin i_1}{\mu_1 \alpha_0 + \mu_0 \alpha_1} = \mathrm{M}$, the equations become, dividing out common factors, such as $1 - \frac{1}{2}\delta^2 \mu_0^2 \cdot \cos^2 i_0 + \frac{1}{2}\delta^2 (\mathrm{C} - \frac{1}{2}\mu_0^2)$, and neglecting $\delta^3, \ldots$ as before,

$$r \,.\, \{\cot .\, (i_0 - i_1) - \iota \mathrm{M}\}\, (1 - \iota \delta \mu_0 \cos i_0)$$

$$\left[\begin{aligned} &1 - \frac{\iota \delta \nu (\mathrm{A} - \mu_0^2)}{\mu_0 \mu_1 \sin (i_0 + i_1)} - \delta^2 \frac{\sin (i_0 - i_1)}{\sin (i_0 + i_1)} (\mathrm{C} - \tfrac{1}{2}\mu_0^2) \\ &\qquad - \iota \delta^2 \frac{\{(\mathrm{A} - \mu_0^2)(\mathrm{A} - \mu_1^2) + (\mu_1^2 - \mu_0^2)(\mathrm{B} - \mathrm{C})\} \,.\, \sin i_0 \sin i_1 \,.\, \epsilon^{-\iota (i_0 + i_1)}}{(\mu_1^2 + \mu_0^2) \sin (i_0 + i_1) \{\cos (i_0 - i_1) - \iota \mathrm{M} \sin (i_0 - i_1)\}} \\ &- \delta (\mathrm{A} - \mu_0^2) \frac{\mu_0 \mu_1}{\mu_1^2 + \mu_0^2} \cdot \frac{\cos (i_0 - i_1)(\sin i_0 \sin i_1 - \alpha_0 \alpha_1) - \iota \sin (i_0 - i_1)(\alpha_0 \sin i_1 - \alpha_1 \sin i_0)}{\nu \,.\, \{\cos (i_0 - i_1) - \iota \mathrm{M} \sin (i_0 - i_1)\}} \\ &- \tfrac{1}{2}\delta \,.\, \left(\mathrm{A}' - \frac{1}{m_0^2}\right) \frac{\mu_0}{\sin i_0} \cdot \mathrm{M} \frac{\epsilon^{\iota (i_0 - i_1)}}{\cos (i_0 - i_1) - \iota \mathrm{M} \sin (i_0 - i_1)} \end{aligned}\right]$$

$$= - (\cot (i_0 + i_1) + \iota \mathrm{M})\, (1 + \iota \delta \mu_0 \cos i_0)$$

$$\left[\begin{aligned} &1 - \frac{\iota \delta \nu (\mathrm{A} - \mu_0^2)}{\mu_0 \mu_1 \sin (i_0 - i_1)} - \delta^2 \frac{\sin (i_0 + i_1)}{\sin (i_0 - i_1)} (\mathrm{C} - \tfrac{1}{2}\mu_0^2) \\ &\qquad - \iota \delta^2 \frac{\{(\mathrm{A} - \mu_0^2)(\mathrm{A} - \mu_1^2) + (\mu_1^2 - \mu_0^2)(\mathrm{B} - \mathrm{C})\} \sin i_0 \sin i_1 \,.\, \epsilon^{\iota (i_0 - i_1)}}{(\mu_1^2 + \mu_0^2) \sin (i_0 - i_1) \{\cos (i_0 + i_1) + \iota \mathrm{M} \sin (i_0 + i_1)\}} \\ &- \delta (\mathrm{A} - \mu_0^2 \frac{\mu_0 \mu_1}{\mu_1^2 + \mu_0^2} \cdot \frac{\cos (i_0 + i_1)(\sin i_0 \sin i_1 - \alpha_0 \alpha_1) + \iota \sin (i_0 + i_1)(\alpha_0 \sin i_1 - \alpha_1 \sin i_0)}{\nu \,.\, \{\cos (i_0 + i_1) + \iota \mathrm{M} \sin (i_0 + i_1)\}} \\ &- \tfrac{1}{2}\delta \left(\mathrm{A}' - \frac{1}{m_0^2}\right) \frac{\mu_0}{\sin i_0} \cdot \mathrm{M} \frac{\epsilon^{-\iota (i_0 + i_1)}}{\cos (i_0 + i_1) + \iota \mathrm{M} \sin (i_0 + i_1)} \end{aligned}\right]$$

and

$$s \,.\, \{\cot (i_0 - i_1) - \iota \mathrm{M}\}\, (1 - \iota \delta \mu_0 \cos i_0 - \tfrac{1}{2}\delta^2 \mu_0^2 \cos^2 i_0) \left[\frac{\text{Bracket}}{\text{of } r}\right]$$

$$= \frac{2 \cos i_0 \sin i_1}{\sin (i_0 - i_1) \sin (i_0 + i_1)} \left[1 - \delta (\mathrm{A} - \mu_0^2) \frac{\mu_0 \mu_1}{\mu_1^2 + \mu_0^2} \cdot \frac{\sin i_0 \sin i_1 - \alpha_0 \alpha_1}{v} - \tfrac{1}{2}\delta \left(\mathrm{A}' - \frac{1}{m_0^2}\right) \frac{\mu_0}{\sin i_0} \cdot \mathrm{M}\right].$$

Now $r = \mathrm{R}\epsilon^{\iota \rho}$, $s = \mathrm{S}\epsilon^{\iota \sigma}$; hence changing ι into $- \iota$, multiplying and dividing corresponding equations, and, as before, neglecting δ^3, $\delta^2 \frac{1}{m^2}$ (and, therefore, $\delta^2 (\sin i_0 \sin i_1 - \alpha_0 \alpha_1)$), we find

$$
\begin{aligned}
\mathrm{R}^2 . [\cot^2 (i_0 - i_1) + \mathrm{M}^2] . \Bigg[& 1 + \delta^2 \frac{(\mathrm{A} - \mu_0^2)(\mathrm{A} - \mu_1^2) + (\mu_1^2 - \mu_0^2)(\mathrm{B} - \mathrm{C})}{\mu_0 \mu_1 (\mu_1^2 + \mu_0^2)^2} \\
& \quad . \frac{\sin i_0 \sin i_1 . \{(\mu_1^2 - \mu_0^2)^2 + 4\mu_0^2\mu_1^2 \cos 2i_0 \cos 2i_1\}}{\sin^2 (i_0 + i_1) \{\cos^2 (i_0 - i_1) + \mathrm{M}^2 \sin^2 (i_0 - i_1)\}} \\
& - 2\,\delta (\mathrm{A} - \mu_0^2) \frac{\mu_1 \mu_0}{\mu_1^2 + \mu_0^2} \\
& \quad . \frac{\cos^2 (i_0 - i_1)(\sin i_0 \sin i_1 - \alpha_0 \alpha_1) + \mathrm{M} \sin^2 (i_0 - i_1)(\alpha_0 \sin i_1 - \alpha_1 \sin i_0)}{\nu . \{\cos^2 (i_0 - i_1) + \mathrm{M}^2 \sin^2 (i_0 - i_1)\}} \\
& - \delta . \left(\mathrm{A}' - \frac{1}{m_0^2}\right) \frac{\mu_0}{\sin i_0} . \mathrm{M} \frac{\cos^2 (i_0 - i_1) - \mathrm{M} \sin^2 (i_0 - i_1)}{\cos^2 (i_0 - i_1) + \mathrm{M}^2 \sin^2 (i_0 - i_1)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= [\cot^2 (i_0 + i_1) + \mathrm{M}^2] . \Bigg[& 1 + \delta^2 \frac{(\mathrm{A} - \mu_0^2)(\mathrm{A} - \mu_1^2) + (\mu_1^2 - \mu_0^2)(\mathrm{B} - \mathrm{C})}{\mu_0 \mu_1 (\mu_1^2 + \mu_0^2)^2} \\
& \quad \frac{\sin i_0 \sin i_1 \{(\mu_1^2 - \mu_0^2)^2 + 4\mu_0^2\mu_1^2 \cos 2i_0 \cos 2i_1\}}{\sin^2 (i_0 - i_1) \{\cos^2 (i_0 + i_1) + \mathrm{M}^2 \sin^2 (i_0 + i_1)\}} \\
& - 2\,\delta (\mathrm{A} - \mu_0^2) \frac{\mu_0 \mu_1}{\mu_1^2 + \mu_0^2} \\
& \quad . \frac{\cos^2 (i_0 + i_1)(\sin i_0 \sin i_1 + \alpha_0 \alpha_1) + \mathrm{M} \sin^2 (i_0 + i_1)(\alpha_0 \sin i_1 - \alpha_1 \sin i_0)}{\nu . \{\cos^2 (i_0 + i_1) + \mathrm{M}^2 \sin^2 (i_0 + i_1)\}} \\
& - \delta . \left(\mathrm{A}' - \frac{1}{m_0^2}\right) \frac{\mu_0}{\sin i_0} . \mathrm{M} . \frac{\cos^2 (i_0 + i_1) - \mathrm{M} \sin^2 (i_0 + i_1)}{\cos^2 (i_0 + i_1) + \mathrm{M}^2 \sin^2 (i_0 + i_1)} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{S}^2 . [\cot^2 (i_0 - i_1) + \mathrm{M}^2] & \left[\begin{matrix}\text{Bracket} \\ \text{of } \mathrm{R}^2\end{matrix}\right] \\
= \frac{4 \cos^2 i_0 \sin^2 i_1}{\sin^2 (i_0 - i_1) \sin^2 (i_0 + i_1)} & \Bigg[1 - 2\delta (\mathrm{A} - \mu_0^2) \frac{\mu_0 \mu_1}{\mu_1^2 + \mu_0^2} \frac{\sin i_0 \sin i_1 - \alpha_0 \alpha_1}{\nu} \\
& - \delta \left(\mathrm{A}' - \frac{1}{m_0^2}\right) \frac{\mu_0}{\sin i_0} . \mathrm{M} \Bigg].
\end{aligned}
$$

$$
\epsilon^{2\iota\rho} = \frac{[\cot (i_0 + i_1) + \iota\mathrm{M}][\cot (i_0 - i_1) + \iota\mathrm{M}]}{[\cot (i_0 + i_1) - \iota\mathrm{M}][\cot (i_0 - i_0) - \iota\mathrm{M}]} . \frac{1 + 2\iota\,\delta\mu_0 \cos i_0}{1 - 2\iota\,\delta\mu_0 \cos i_0} \; \frac{1 - 2\iota\,\delta\mu_0 \cos i_0 \, \dfrac{\mathrm{A} - \mu_0^2}{\mu_1^2 - \mu_0^2}}{1 + 2\iota\,\delta\mu_0 \cos i_0 \, \dfrac{\mathrm{A} - \mu_0^2}{\mu_1^2 - \mu_0^2}}
$$

$$
= \frac{\left\{1 + \iota\mathrm{M} . \dfrac{\tan (i_0 - i_1) + \tan (i_0 + i_1)}{1 - \mathrm{M}^2 \tan (i_0 - i_1) \tan (i_0 + i_1)}\right\} . \left\{1 + 2\iota\,\delta\mu_0 \cos i_0 \dfrac{\mu_1^2 - \mathrm{A}}{\mu_1^2 - \mu_0^2}\right\}}{\left\{1 - \iota\mathrm{M} . \dfrac{\tan (i_0 - i_1) + \tan (i_0 + i_1)}{1 - \mathrm{M}^2 \tan (i_0 - i_1) \tan (i_0 + i_1)}\right\} \left\{1 - 2\iota\,\delta\mu_0 \cos i_0 \dfrac{\mu_1^2 - \mathrm{A}}{\mu_1^2 - \mu_0^2}\right\}} .
$$

$$
\epsilon^{2\iota\sigma} = \frac{1 + \iota\mathrm{M} \tan (i_0 - i_1)}{1 - \iota\mathrm{M} \tan (i_0 - i_1)} \; \frac{1 + \iota\,\delta \dfrac{\mathrm{A} - \mu_0^2}{\mu_0 \mu_1} \dfrac{\nu}{\sin (i_0 + i_1)}}{1 - \iota\,\delta \dfrac{\mathrm{A} - \mu_0^2}{\mu_0 \mu_1} \dfrac{\nu}{\sin (i_0 + i_1)}} .
$$

Hence, we have finally, using the values of α_0, α_1,

$$
\left.
\begin{aligned}
R^2 &= \frac{\cot^2(i_0+i_1)+M^2}{\cot^2(i_0-i_1)+M^2}\\
&\cdot\Bigg[1+\delta^2\frac{(A-\mu_0^2)(A-\mu_1^2)+(\mu_1^2-\mu_0^2)(B-C)}{\mu_0\mu_1(\mu_1^2+\mu_0^2)^2}\\
&\qquad\frac{\sin i_0\sin i_1\sin 2i_0\sin 2i_1\{(\mu_1^2-\mu_0^2)^2+4\mu_0^2\mu_1^2\cos 2i_0\cos 2i_1\}}{\sin^2(i_0-i_1)\sin^2(i_0+i_1)\{\cos^2(i_0-i_1)+M^2(i_0-i_1)\}\{\cos^2(i_0+i_1)+M^2\sin^2(i_0+i_1)\}}\\
&\qquad+2\delta\,.\,\frac{\mu_0^2\mu_1^2(\mu_1^2-\mu_0^2)}{(\mu_1^2+\mu_0^2)^3}\\
&\qquad\frac{\sin 2i_0\sin 2i_1\,.\left\{A'(\mu_1^2-\mu_0^2)+(A-\mu_1^2)\dfrac{1}{m_0^2}-(A-\mu_0^2)\dfrac{1}{m_1^2}\right\}}{\mu_0\sin i_0\,.\{\cos^2(i_0-i_1)+M^2\sin^2(i_0-i_1)\}\{\cos^2(i_0+i_1)+M^2\sin^2(i_0+i_1)\}}\Bigg]\\[2ex]
S^2 &= \frac{4\cos^2 i_0\,.\sin^2 i_1}{\sin^2(i_0-i_1)\,.\sin^2(i_0+i_1)\{\cot^2(i_0-i_1)+M^2\}}\\
&\cdot\Bigg[1-(\delta^2)\frac{(A-\mu_0^2)(A-\mu_1^2)+(\mu_1^2-\mu_0^2)(B-C)}{\mu_0\mu_1(\mu_1^2+\mu_0^2)^2}\\
&\qquad\frac{\sin i_0\sin i_1\{(\mu_1^2-\mu_0^2)^2+4\mu_0^{\,2}{}_1\mu^2\cos 2i_0\cos 2i_1\}}{\sin^2(i_0+i_1)\{\cos^2(i_0-i_1)+M^2\sin^2(i_0-i_1)\}}\\
&\qquad-2\delta\frac{\mu_0^2\mu_1^2(\mu_1^2-\mu_0^2)}{(\mu_1^2+\mu_0^2)^3}\cdot\frac{\sin^2(i_0-i_1)\left\{A'(\mu_1^2-\mu_0^2)+(A-\mu_1^2)\dfrac{1}{m_0^2}-(A-\mu_0^2)\dfrac{1}{m_1^2}\right\}}{\mu_0\sin i_0\,.\,\{\cos^2(i_0-i_1)+M^2\sin^2(i_0-i_1)\}}\Bigg]\\
&\text{and}\\
\tan\rho &= M\frac{\tan(i_0-i_1)+\tan(i_0+i_1)}{1-M^2\tan(i_0-i_1)\,.\tan(i_0+i_1)}\\
&\qquad+2\delta\mu_0\cos i_0\cdot\frac{\mu_1^2-A}{\mu_1^2-\mu_0^2}\,\frac{\{1+M^2\tan^2(i_0-i_1)\}\{1+M^2\tan^2(i_0+i_1)\}}{\{1-M^2\tan(i_0-i_1)\tan(i_0+i_1)\}^2}\\
\tan\sigma &= M\tan(i_0-i_1)+\delta\mu_0\sin i_0\cdot\frac{A+\mu_0\mu_1\cos(i_0+i_1)}{\mu_0\mu_1\sin(i_0+i_1)}\{1+M^2\tan^2(i_0-i_1)\}
\end{aligned}
\right\}\text{(X.).}
$$

And here

$$
M=\frac{\mu_1\sin i_0-\mu_0\sin i_1}{\mu_0\alpha_1+\mu_1\alpha_0}=\frac{\mu_1^2-\mu_0^2}{\mu_1^2+\mu_0^2}\left\{1+\frac{\mu_0\mu_1}{\mu_1^2+\mu_0^2}\cdot\frac{\dfrac{1}{m_0^2}+\dfrac{1}{m_1^2}}{2\sin i_0\sin i_1}\right\},
$$

as long as $\sin^2 i_0 > 1/m_0^2$, and $\sin^2 i_1 > 1/m_1^2$.

These give, when $i_0 = 0$,

$$R^2 = \left(\frac{\mu_1 - \mu_0}{\mu_1 + \mu_0}\right)^2 \cdot \left[1 + 4\,\delta^2 \mu_0 \mu_1 \frac{(A - \mu_0^2)(A - \mu_1^2) + (\mu_1^2 - \mu_1^2)(B - C)}{(\mu_1^2 - \mu_0^2)^2}\right],$$

$$S^2 = \frac{4\mu_0^2}{(\mu_1 + \mu_0)^2}\left[1 - \delta^2 \cdot \frac{(A - \mu_0^2)(A - \mu_1^2) + (\mu_1^2 - \mu_0^2)(B - C)}{(\mu_1 + \mu_0)^2}\right],$$

the same as for vibrations parallel to the plane of incidence, as should be the case.

At the polarizing angle, when $(i_0 + i_1) = \frac{\pi}{2}$, we have, since then

$$M = \frac{\mu_1^2 - \mu_0^2}{\mu_1^2 + \mu_0^2}(1 + 1/2m_0^2 + 1/2m_1^2),$$

$$\begin{aligned} R^2 = {} & 4\delta^2 . \mu_0^2 \mu_1^2 . (\mu_1^2 + \mu_0^2)(\mu_1^2 - \mu_0^2)^4 \frac{(A - \mu_0^2)(A - \mu_1^2) + (\mu_1^2 - \mu_0^2)(B - C)}{\mu_0^3 + 14\mu_0^4 \mu_1^4 + \mu_1^8)^2} \\ & + 8\mu_0^3 \mu_1^3 . (\mu_1^2 - \mu_0^2)^3 . \sqrt{\mu_1^2 + \mu_0^2}\, \frac{A'(\mu_1^2 - \mu_0^2) + (A - \mu_1^2)\frac{1}{m_0^2} - (A - \mu_0^2)\frac{1}{m_0^2}}{(\mu_0^8 + 14\mu_0^4 \mu_1^4 + \mu_1^8)^2} \end{aligned}$$

$$\begin{aligned} S^2 = {} & \frac{4\mu_0^4 . (\mu_1^2 + \mu_0^2)^2}{\mu_0^8 + 14\mu_0^4 \mu_1^4 + \mu_1^8}\left[1 - \delta^2 . \frac{(\mu_1^2 - \mu_0^2)^4}{\mu_1^2 + \mu_0^2} \frac{(A - \mu_0^2)(A - \mu_1^2) + (\mu_1^2 - \mu_0^2)(B - C)}{\mu_0^8 + 14\mu_0^4 \mu_1^4 + \mu_1^8}\right. \\ & \left. - 2\delta \frac{\mu_0 \mu_1 (\mu_1^2 - \mu_0^2)^3}{(\mu_1^2 + \mu_0^2)^{\frac{3}{2}}} \cdot \frac{A'(\mu_1^2 - \mu_0^2) + (A - \mu_1^2)\frac{1}{m_0^2} - (A - \mu_0^2)\frac{1}{m_1^2}}{\mu_0^8 + 14\mu_0^4 \mu_1^4 + \mu_1^8}\right] \end{aligned}$$

$$\begin{aligned} \tan\rho = {} & -\frac{2\mu_0 \mu_1 (\mu_1^2 + \mu_0^2)}{(\mu_1^2 - \mu_0^2)^2}(1 - 1/2m_0^2 - 1/2m_1^2) \\ & + 2\delta \frac{\mu_0{}^2 . (\mu_0^8 + 14\mu_1^4 \mu_0^4 + \mu_1^8)(\mu_1^2 - A)}{\sqrt{\mu_1^2 + \mu_0^2} . (\mu^2{}_1 - \mu_0^2)^5}\left\{1 - \frac{4\mu_0^2 \mu_1^2 (\mu_1^2 + \mu_1^2)^2}{\mu_0^8 + 14\mu_0^4 \mu_1^4 + \mu_1^8}(1/m_0^2 + 1/m_1^2)\right\}. \end{aligned}$$

§ 6. *Summary of Results.*

We shall shortly summarize those results that are of use for comparing with experiment.

Vibrations perpendicular to plane of incidence. Plane of polarization parallel to plane of incidence.

These give the sine-formula of FRESNEL, which holds for parallel polarized light.

$$\left.\begin{aligned} (R\,||)^2 &= \frac{\sin^2(i_0 - i_1)}{\sin^2(i_0 + i_1)}\left[1 + 4\,\delta^2 . \mu_0 \mu_1 \cos i_0 \cos i_1 \frac{(A - \mu_0^2)(A - \mu_1^2) + (B - C)(\mu_1^2 - \mu_0^2)}{(\mu_1^2 - \mu_0)^2}\right] \\ \tan(\rho\,||) &= 2\,\delta . \mu_0 \cos i_0 . \frac{\mu_1^2 - A}{\mu_1^2 - \mu_0^2} \quad \ldots\ldots\ldots\ldots\ldots \end{aligned}\right\} \text{(VIII., p. 840).}$$

Vibrations parallel to plane of incidence. Plane of polarization perpendicular to plane of incidence.

These correspond to FRESNEL'S tangent formula for perpendicularly polarized light.

Electromagnetic and Contractile Ether Theories.

$$\left.\begin{aligned}
(\mathrm{R}\perp)^2 = \frac{\tan^2(i_0-i_1)}{\tan^2(i_0+i_1)}\Bigg[& 1+4\,\delta^2\,.\,\mu_0\mu_1\cos i_0\cos i_1\,\frac{\mathrm{B}-\mathrm{C}-\mathrm{J}\,\mu_0\mu_1\sin i_0\sin i_1}{\mu_1{}^2\cos^2 i_0-\mu_0{}^2\cos^2 i_1} \\
& +4\,\delta^2\mu_0\mu_1\cos i_0\cos i_1\,\frac{\{\mathrm{A}-\mu_0{}^2-(\mathrm{A}-\mathrm{G}\mu_0{}^4)\sin^2 i_0\}\{\mathrm{A}-\mu_1{}^2-(\mathrm{A}-\mathrm{G}\mu_1{}^2)\sin^2 i_1\}}{(\mu_1{}^2\cos^2 i_0-\mu_0{}^2\cos^2 i_1)^2}\Bigg] \\
\tan(\rho\perp) = 2\,\delta\,.\,\mu_0\cos i_0\,.\,& \frac{\mu_1{}^2-\mathrm{A}+(\mathrm{A}-\mathrm{G}\mu_1{}^4)\sin^2 i_1}{\mu_1{}^2\cos^2 i_0-\mu_0{}^2\cos^2 i_1} \quad . \quad .
\end{aligned}\right\}\text{(IX., p. 843).}$$

Elastic Solid Theory.

$$\left.\begin{aligned}
(\mathrm{R}\perp)^2 = {} & \frac{\cot^2(i_0+i_1)+\mathrm{M}^2}{\cot^2(i_0-i_1)+\mathrm{M}^2} \\
& .\Bigg[1+\delta^2\,\frac{(\mathrm{A}-\mu_0{}^2)(\mathrm{A}-\mu_1{}^2)+(\mu_1{}^2-\mu_0{}^2)(\mathrm{B}-\mathrm{C})}{\mu_0\mu_1(\mu_1{}^2+\mu_0{}^2)^2} \\
& \quad .\,\frac{\sin i_0\sin i_1\,.\,\sin 2i_0\sin 2i_1\,.\,\{(\mu_1{}^2-\mu_0{}^2)^2+4\mu_0{}^2\mu_1{}^2\cos 2i_0\cos 2i_1\}}{\sin^2(i_0-i_1)\sin^2(i_0+i_1)\,.\,\{\cos^2(i_0-i_1)+\mathrm{M}^2\sin^2(i_0-i_1)\}\{\cos^2(i_0+i_1)+\mathrm{M}^2\sin^2(i_0+i_1)\}} \\
& \quad +2\,\delta\,.\,\frac{\mu_0{}^2\mu_1{}^2(\mu_1{}^2-\mu_0{}^2)}{(\mu_0{}^2+\mu_0{}^2)^3} \\
& \quad .\,\frac{\sin 2i_0\sin 2i_1\,.\left\{\mathrm{A}'\,.\,(\mu_1{}^2-\mu_0{}^2)+(\mathrm{A}-\mu_1{}^2)\,\dfrac{1}{m_0{}^2}-(\mathrm{A}-\mu_0{}^2)\,\dfrac{1}{m_1{}^2}\right\}}{\mu_0\sin i_0\,.\,\{\cos^2(i_0-i_1)+\mathrm{M}^2\sin^2(i_0-i_1)\}\{\cos^2(i_0+i_1)+\mathrm{M}^2\sin^2(i_0+i_1)\}}\Bigg] \\
\tan(\rho\perp) = {} & \mathrm{M}\,\frac{\tan(i_0-i_1)+\tan(i_0+i_1)}{1-\mathrm{M}^2\tan(i_0-i_1)\tan(i_0+i_1)} \\
& +2\,\delta\mu_0\cos i_0\,\frac{\mu_1{}^2-\mathrm{A}}{\mu_1{}^2-\mu_0{}^2}\,\frac{\{1+\mathrm{M}^2\tan^2(i_0-i_1)\}\{1+\mathrm{M}^2\tan^2(i_0+i_1)\}}{\{1-\mathrm{M}^2\tan(i_0-i_1)\tan(i_0+i_1)\}^2}. \quad .
\end{aligned}\right\}\text{(X., p. 852).}$$

where

$$\mathrm{M} = \frac{\mu_1\sin i_0-\mu_0\sin i_1}{\mu_0\sqrt{\sin^2 i_0-\dfrac{1}{m_0{}^2}}+\mu_1\sqrt{\sin^2 i_1-\dfrac{1}{m_1{}^2}}} = \frac{\mu_1{}^2-\mu_0{}^2}{\mu_1{}^2+\mu_0{}^2}\left\{1+\frac{\mu_0\mu_1}{\mu_1{}^2+\mu_0{}^2}\,\frac{\dfrac{1}{m_0{}^2}+\dfrac{1}{m_1{}^2}}{2\sin i_0\sin i_1}\right\}$$

provided $\sin^2 i_0 > \dfrac{1}{m_0{}^2}$, $\sin^2 i_1 > \dfrac{1}{m_1{}^2}$.

There is a point here which calls for remark, viz., as to the quadrant in which $\rho\perp$, $\rho \parallel$ are to be taken. Neglecting δ, in the equations (V′.) of § 5, p. 839, we find the large part of r to be $-\frac{\sin(i_0 - i_1)}{\sin(i_0 + i_1)}$, which is negative, so that for parallel-polarized light at normal incidence the vibration in the reflected light is opposite to that in the incident at the reflecting surface, so that there is a retardation of phase $= \pi$, Similarly, the equations (VI′.) give for the important part of r for perpendicularly-polarized light $-\frac{\tan(i_0 - i_1)}{\tan(i_0 + i_1)}$, which is negative at normal incidence, but positive for incidences greater than the polarizing angle.

We shall suppose $R \parallel$, $R \perp$ to be taken equal to the absolute values of the above ratios. Then $\rho \parallel$ will lie between π and 2π, or between π and 0, according as $\tan(\rho \parallel)$ is + or −, and will differ from π by an amount of the order δ. The same will apply to $\rho \perp$, whose difference from π, however, does not remain of order δ, but which increases through $3\pi/2$ to 2π, or decreases through $\frac{1}{2}\pi$ to 0, according to the sign of $\tan(\rho \perp)$.

The difference $\rho \perp - \rho \parallel$—the retardation of phase of the perpendicularly- over the parallel-polarized light—is positive or negative according as $\tan(\rho \perp)$ is positive or negative, and increases numerically from 0 at normal incidence through $\pm \frac{1}{2}\pi$ at the polarizing angle to $\pm \pi$ at grazing incidence. And the reflection is said by JAMIN to be positive or negative as the case may be.

If a ray of elliptically-polarized light be reflected normally from a surface, then the difference of phase of the components, and the position of the axes of the vibrational ellipse, as well as the direction of its description, are all unchanged in space, but with reference to the direction of propagation, and, therefore, also to an observer viewing both rays, the position of the axes has changed into one symmetrical to the former one, with respect to the plane of incidence, and the ray from being right-handed has become left-handed, or *vice versâ*. Thus, there is an apparent change of phase of π, which is called by JAMIN "π de retournement," and causes him to give the measured difference of phase as lying between π and 2π, instead of between 0 and π.

We must also consider the effect of a finite, though large, velocity for the pressural wave in the Elastic Solid Theory. We have made no supposition as to the values of the m's, the ratios of the pressural-wave velocity to that of light in the different media, except that these ratios are large. The ratio $m_0 : m_1$ may have any value, so that the refractive index for the pressural-wave between the two media may also have any value. The effect of the pressural-wave is to add to $\frac{\mu_1^2 - \mu_0^2}{\mu_1^2 + \mu_0^2}$ a quantity $\frac{(\mu_1^2 - \mu_0^2)\,\mu_0\mu_1}{2\,(\mu_0^2 + \mu_1^2)^2} \frac{\frac{1}{m_0^2} + \frac{1}{m_1^2}}{\sin i_0 \sin i_1}$, for moderately-large values of i_0, such as are used in most of the experiments; in $(R \perp)^2$, also, there is an additional term, which at no angle of incidence is of magnitude more than comparable with $1/m^2$.

Now m^2 is large, perhaps 100, as above, § 3, p. 834. The term in $(R\perp)^2$ may always be neglected; and at all but very small angles of incidence M be put equal to

$\frac{\mu_1^2 - \mu_0^2}{\mu_1^2 + \mu_0^2}$. This result is in agreement with GREEN and in opposition to HAUGHTON, who proposes to make $M = \frac{n^2 - 1}{n^2 + 1}$, where $n < \frac{\mu_1}{\mu_0}$, ascribing it to a difference between the refractive index for the pressural-wave and that for light; but it has been shown above that no such difference could diminish n and therefore M.

The formulæ VIII., IX., and X. can be put into a more suitable form for calculation; the quantities experimentally determined are usually $\frac{R\perp}{R\parallel}$ and $\rho\perp - \rho\parallel$. In doing so we neglect powers of δ above the second and make use of SNELL's law $\mu_0 \sin i_0 = \mu_1 \sin i_1$ and the equations $\sin(i_0 - i_1)\sin(i_0 + i_1) = \frac{\mu_1^2 - \mu_0^2}{\mu_0\mu_1}\sin i_0 \sin i_1$, and $\cos(i_0 - i_1)\cos(i_0 + i_1) = 1 - \frac{\mu_1^2 + \mu_0^2}{\mu_0\mu_1}\sin i_0 \sin i_1$.

With the same notation for the constants of the variable layer as before, viz., d = thickness, $\delta = \frac{2\pi d}{\lambda}$, μ = refractive index, and $A = \frac{1}{d}\int_0^d \mu^2\,dx$ = mean value of μ^2,

$$B - C = \frac{1}{d^2}\int_0^d \mu^2 (2x - d)\,dx,\quad G = \frac{1}{d}\int_0^d \frac{dx}{\mu^2},\quad J = \frac{1}{d^2}\int_0^d\!\!\int_0^x \left(\frac{\mu_x^2}{\mu_\xi^2} - \frac{\mu_\xi^2}{\mu_x^2}\right) d\xi\,dx,$$

since these enter into the expressions in different combinations, we shall introduce a different set of constants, involving A, B − C, G, J, and d together with μ_0, μ_1, and defined by the following equations—

$$A = \frac{4\{A(\mu_1^2 + \mu_0^2) - 2\mu_0^2\mu_1^2\}(A - \mu_0^2 - \mu_1^2 + G\mu_0^2\mu_1^2) + \{(B - C)(\mu_1^2 + \mu_0^2) - J\mu_0^2\mu_1^2\}(\mu_1^2 - \mu_0^2)}{(\mu_1^2 - \mu_0^2)^2} \cdot \left(\frac{2\pi d}{\lambda}\right)^2,$$

$$B = 4\mu_0\mu_1 \frac{(A - \mu_0^2 - \mu_1^2 + G\mu_0^2\mu_1^2)^2}{(\mu_1^2 - \mu_1^2)^2} \cdot \left(\frac{2\pi d}{\lambda}\right)^2,$$

$$C = 64\,\frac{\mu_0^3\mu_1^3 \cdot \{(A - \mu_0^2)(A - \mu_1^2) + (\mu_1^2 - \mu^2)(B - C)\}}{(\mu_1^2 + \mu_0^2)^4}\left(\frac{2\pi d}{\lambda}\right)^2,$$

$$D = 2\mu_0 \frac{\mu_1^2 - A}{\mu_1^2 - \mu_0^2} \cdot \frac{2\pi d}{\lambda},$$

$$E = -2\mu_0 \frac{A - \mu_0^2 - \mu_1^2 + G\mu_0^2\mu_1^2}{\mu_1^2 - \mu_0^2} \cdot \frac{2\pi d}{\lambda}.$$

Then the expressions for $\frac{R\perp}{R\parallel}$, and $\rho\perp - \rho\parallel$ become—

Electromagnetic and Contractile Ether Theories :—

$$\left.\begin{aligned}
\left(\frac{\mathrm{R}\perp}{\mathrm{R}\|}\right)^2 &= \frac{\cos^2(i_0+i_1)}{\cos^2(i_0-i_1)}\left[1 + \mathrm{A}\,\frac{\sin i_0 \sin i_1 \cos i_0 \cos i_1}{\cos(i_0-i_1)\cos(i_0+i_1)} + \mathrm{B}\,\frac{\sin^2 i_0 \sin^2 i_1 \cos i_0 \cos i_1}{\cos^2(i_0-i_1)\cos^2(i_0+i_1)}\right]\\
\tan(\rho\|) &= \mathrm{D}\cos i_0\\
\tan(\rho\perp) - \tan(\rho\|) &= \mathrm{E}\,\frac{\sin^2 i_0 \cos i_0}{\cos(i_0-i_1)\cos(i_0+i_1)}\\
\tan(\rho\perp - \rho\|) &= \frac{\mathrm{E}\sin^2 i_0 \cos i_0}{\cos(i_0-i_1)\cos(i_0+i_1)+\mathrm{F}}\;{}^{*}
\end{aligned}\right\}\quad \text{XI.}$$

These expressions are true as far as order d^2/λ^2, provided $\lambda/2\pi d >$ greatest value of μ occurring in the variable layer.

Except in the neighbourhood of the polarizing angle, $\tan(\rho\perp - \rho\|)$ reduces to

$$\frac{\mathrm{E}\sin^2 i_0 \cos i_0}{\cos(i_0-i_1)\cos(i_0+i_1)}.$$

Elastic Solid Theory.

Here we introduce subsidiary angles defined by the equations

$$\tan\alpha = \mathrm{M}\tan(i_0+i_1) \quad \tan\beta = \mathrm{M}\tan(i_0-i_1), \qquad \text{where } \mathrm{M} = \frac{\mu_1^2-\mu_0^2}{\mu_1^2+\mu_0^2}.$$

Then we have

$$\left.\begin{aligned}
\left(\frac{\mathrm{R}\perp}{\mathrm{R}\|}\right)^2 &= \frac{\cos^2\beta.\cos^2(i_0+i_1)}{\cos^2\alpha.\cos^2(i_0-i_1)}\left[1 - \mathrm{C}\,\frac{\cos^2\alpha.\cos^2\beta.\sin^2 i_0 \sin^2 i_1 \cos i_0 \cos i_1}{\cos^2(i_0-i_1)\cos^2(i_0+i_1)}\right]\\
\tan(\rho\|) &= \mathrm{D}.\cos i_0\\
\tan(\rho\perp) - \tan(\rho\|) &= \tan(\alpha+\beta).[1 + \mathrm{D}\cos i_0.\tan(\alpha+\beta)]\\
\tan(\rho\perp - \rho\|) &= \frac{\cot(\alpha+\beta)+\mathrm{D}\cos i_0}{\cot^2(\alpha+\beta)(1+\mathrm{D}^2\cos^2 i_0)+\mathrm{D}\cos i_0\cot(\alpha+\beta)+\mathrm{D}^2\cos^2 i_0}\\
\text{or}\qquad&\\
\cot(\rho\perp - \rho\|) &= \cot(\alpha+\beta) + \mathrm{D}^2.\frac{\cos^2 i_0.\operatorname{cosec}^2(\alpha+\beta)}{\cot(\alpha+\beta)+\mathrm{D}\cos i_0}
\end{aligned}\right\}.\quad \text{(XII.)}$$

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* The expression for $\tan(\rho\perp - \rho\|)$ inclusive of terms involving $(2\pi d/\lambda)^3$ is of the form

$$\frac{\mathrm{E}\sin^2 i_0 \cos i_0}{\cos(i_0-i_1)\cos(i_0+i_1)(1+a\sin^2 i_0+b\sin^4 i_0+\ldots)+a'+b'\sin^2 i_0+\ldots},$$

$a, b, \ldots a', b', \ldots$ being constants of order $(2\pi d/\lambda)^2$. Since $\tan(\rho\perp - \rho\|)$ is large only in the neighbourhood of the polarizing angle I, we may put $i_0 = \mathrm{I}$ in the small terms, thus obtaining the expression in the text. Then

$$\mathrm{E} = \frac{-2\mu_0\,\dfrac{\mathrm{A}-\mu_0^2-\mu_1^2+\mathrm{G}\mu_0^2\mu_1^2}{\mu_1^2-\mu_0^2}}{1+a\sin^2\mathrm{I}+b\sin^4\mathrm{I}+\ldots}\,\frac{2\pi d}{\lambda}, \qquad \mathrm{F} = \frac{a'+b'\sin^2\mathrm{I}+\ldots}{1+a\sin^2\mathrm{I}+b\sin^4\mathrm{I}+\ldots}.$$

These expressions are true as far as order d^2/λ^2, provided $\frac{9}{11}\lambda/2\pi d >$ greatest value of μ.

To get some idea of the limiting thicknesses of the film, let us compare them with soap-films; REINOLD and RÜCKER estimate the thickness of a black soap-film at about $\cdot 117 \times 10^{-5}$ centim., that of a film showing red of the 1st order at about $2\cdot 84 \times 10^{-5}$ centim. Hence for

black soap-film $\lambda/2\pi d$ is, for line	A	10,	D	8,	H	6
red of 1st order		$\frac{2}{5}$,		$\frac{1}{3}$,		$\frac{1}{4}$

Since the refractive indices of transparent substances lie between 1 and 3, it follows that a transition layer to which the above analysis is to be applicable must certainly be less than that necessary to show even a red of the 1st order.

§ 7. *Comparison of Theory with Experiment—Elastic Solid Theory.*

The expression found for the change of phase is by (XII.)—

$$\tan(\rho\perp - \rho\,\|) = \frac{\cot(\alpha+\beta) + \mathrm{D}\cos i_0}{\cot^2(\alpha+\beta)\,(1+\mathrm{D}^2\cos^2 i_0) + \mathrm{D}\cos i_0\cot(\alpha+\beta) + \mathrm{D}^2\cos^2 i_0}$$

where $\tan\alpha = \mathrm{M}\tan(i_0 + i_1)$, $\tan\beta = \mathrm{M}\tan(i_0 - i_1)$, $\mathrm{M} = \dfrac{\mu_1^2 - \mu_0^2}{\mu_1^2 + \mu_0^2}$, and D is a disposable constant.

The denominator of $\tan(\rho\perp - \rho\,\|)$ may be written $\mathrm{D}^2\cos^2 i_0\,[\frac{3}{4} + \cot^2(\alpha+\beta)] + [\cot(\alpha+\beta) + \frac{1}{2}\mathrm{D}\cos i_0]^2$ and this cannot vanish even to order D^2 unless $\alpha + \beta = \frac{1}{2}\pi$.

Now, $\alpha + \beta = \frac{1}{2}\pi$ gives $\cot(i_0 + i_1)\cot(i_0 - i_1) = \mathrm{M}^2$, or $\dfrac{1 - \sin^2 i_0 - \sin^2 i_1}{\sin^2 i_0 - \sin^2 i_1} = \mathrm{M}^2$, whence $i_0 = \sin^{-1}.\,\dfrac{\frac{1}{2}(\mu_1^2 + \mu_0^2)}{\sqrt{(\mu_1^4 + 3\mu_0^4)}}$ instead of BREWSTER's angle $i_0 = \tan^{-1}\mu_1/\mu_0$. In order that we should obtain BREWSTER's angle it is necessary that M should be only a small fraction ϵ of $\dfrac{\mu_1^2 - \mu_0^2}{\mu_1^2 + \mu_0^2}$, which would give $\sin^2 i_0 = \dfrac{\mu_1^2}{\mu_1^2 + \mu_0^2}\,\dfrac{1}{1 + \epsilon^2\left(\dfrac{\mu_1^2 - \mu_0^2}{\mu_1^2 + \mu_0^2}\right)^3}$. This, as is well known, was pointed out by HAUGHTON, who thought it possible that a smaller effective refractive index for the pressural-wave would lead to such a value of M, but the rigid theory developed above, which includes the most general theory possible, according to VOIGT, without absorption, shows that any alteration in the refractive index for the pressural-waves consistent with keeping their velocity of propagation large could only produce a very slight change in the value of M—and that an increase—except at very small angles of incidence. It is clear then that a rigid Elastic Solid Theory cannot explain the change of phase at reflection.

Electromagnetic and Contractile Ether Theories.

Here the expressions (XI.) for the amplitude and change of phase at reflection contain four constants A, B, E, F, of which the really effective ones are B, E. The constant A in the expression for the ratio of the amplitudes is multiplied by $\cos(i_0 + i_1)$, and thus is without effect at the polarizing angle, at which the deviation from FRESNEL'S formula is most marked. A cannot therefore be determined with any great accuracy, seeing that a considerable change in its value produces only a very slight effect on the result. In some cases it may be put = zero without impairing the accuracy of the formula.

The same considerations apply to the constant F in the expression for the phases.

The other two constants B, E ought to satisfy the condition, $E^2 = \frac{\mu_0}{\mu_1} . B$. As regards accuracy of determination the order of the constants is E, B, F, A.

The experiments discussed are those of JAMIN on solids and liquids (see his two papers, 'Ann. de Chimie et Physique,' série III., 29 (1850) and 31 (1852); a series for flint-glass by KURZ ('Pogg. Ann.,' 108), and some of QUINCKE'S ('Pogg. Ann.,' 128)). Of these the experiments of JAMIN are much the best, and are almost as well represented by the empirical formulæ of CAUCHY as by the theoretical formulæ found above. This might excite surprise—seeing that CAUCHY'S formulæ involve only one independent constant, the ellipticity ϵ—did we not remember that of the three independent constants B, F, A (E of course is not independent), two, F and A, do not have much influence on the result. The experiments of QUINCKE are the most irregular, but they are of interest because QUINCKE investigates the reflection in each other from the bounding surface of pairs of media. Of these I have only taken those in which there are ten or more different determinations, where there is some chance of the constants being accurately determined. The experiments of HAUGHTON ('Phil. Trans.,' 1863) I have not had time to consider, but, with but one or two exceptions, his series consist of too few determinations to allow of an accurate determination of the constants.

In all the above cases measurements were made of the difference of phase, by means of a BABINET'S compensator, directly, and of the ratio of the intensities, indirectly. The polarizer was placed at a large angle α with the plane of incidence, so that in the incident beam the component polarized perpendicularly to the plane of incidence is of great intensity relative to the parallel component. The azimuth β of the reflected light was determined. Then $R\perp/R\parallel$ is given by the equation $R\perp/R\parallel = \tan\varpi = \tan\beta/\tan\alpha$. By this means the determination of ϖ is rendered more accurate, firstly, because the absolute error in ϖ is made much less than that of β owing to the largeness of $\tan\alpha$, and secondly, because the determination of β is itself more accurate, the intensities of the components in the reflected light being more nearly equal.

In combining the experiments I have assumed as a first approximation that the accuracy is the same for all values of δ, the difference of phase, and likewise for all values of β. In strictness this is not true, since the accuracy of the readings is greater the more nearly equal are the intensities of the two components of the reflected light. But as β in most cases ranges from above to below 45°, the assumption will be sufficiently true to give values of the constants not far removed from their most probable values.

The sets of constants A and B, and F and E have in each case been determined independently by making the sum of the squares of the errors in α and β, respectively, a minimum.

We have by (XI.)

$$\tan\delta = \frac{\mathrm{E}\sin^2 i_0 \cos i_0}{\cos(i_0 - i_1)\cos(i_0 + i_1) + \mathrm{F}},$$

$$\tan^2 \varpi = \frac{\cos^2(i_0+i_1)}{\cos^2(i_0-i_1)} + \mathrm{A}\frac{\sin i_0 \sin i_1 \cos i_0 \cos i_1 \cos(i_0+i_1)}{\cos^3(i_0-i_1)} = \mathrm{B}\frac{\sin^2 i_0 \sin^2 i_1 \cos i_0 \cos i_1}{\cos^4(i_0-i_1)} = \frac{\tan^2\beta}{\tan^2\alpha}.$$

Let δ, β be the true, δ', β' the observed values, and let δ_0, β_0 be approximate values, given tentative values A_0, B_0, F_0, E_0 of the constants.

Let $\mathrm{A} = \mathrm{A}_0 + a$, $\mathrm{B} = \mathrm{B}_0 + b$, $\mathrm{F} = \mathrm{F}_0 + f$, $\mathrm{E} = \mathrm{E}_0 + e$, a, b, f, e being small quantities to be determined by the conditions

$$\Sigma(\delta' - \delta)^2 = \text{minimum}, \quad \Sigma(\beta' - \beta)^2 = \text{minimum}.$$

Then, substituting for δ, β their values $\delta_0 + f\frac{\partial\delta_0}{\partial\mathrm{F}_0} + e\frac{\partial\delta_0}{\partial\mathrm{E}_0}$, $\beta_0 + a\frac{\partial\beta_0}{\partial a_0} + b\frac{\partial\beta_0}{\partial\mathrm{B}_0}$, in the equations

$$\Sigma(\delta' - \delta)\frac{\partial\delta}{\partial f} = 0, \quad \Sigma(\delta' - \delta)\frac{\partial\delta}{\partial e} = 0, \quad \Sigma(\beta' - \beta)\,.\,\frac{\partial\beta}{\partial a} = 0, \quad \Sigma(\beta' - \beta)\frac{\partial\beta}{\partial b} = 0,$$

and measuring $\delta' - \delta$ $\beta' - \beta$ in degrees, we obtain, neglecting squares of small quantities—

$$f\Sigma\left(\frac{\partial\delta_0}{\partial\mathrm{F}_0}\right)^2 + e\Sigma\left(\frac{\partial\delta_0}{\partial\mathrm{F}_0}\right)\left(\frac{\partial\delta_0}{\partial\mathrm{E}_0}\right) = \Sigma\frac{\pi(\partial' - \partial_0)}{180}\cdot\left(\frac{\partial\delta_0}{\partial\mathrm{F}_0}\right)$$

$$f\Sigma\left(\frac{\partial\delta_0}{\partial\mathrm{F}_0}\right)\left(\frac{\partial\delta_0}{\partial\mathrm{E}_0}\right) + e\Sigma\left(\frac{\partial\delta_0}{\partial\mathrm{E}_0}\right)^2 = \Sigma\frac{\pi(\partial' - \partial_0)}{180}\cdot\left(\frac{\partial\delta_0}{\partial\mathrm{E}_0}\right)$$

$$a\Sigma\left(\frac{\partial\beta_0}{\partial a_0}\right)^2 + b\Sigma\left(\frac{\partial\beta_0}{\partial a_0}\right)\left(\frac{\partial\beta_0}{\partial\mathrm{B}_0}\right) = \Sigma\frac{\pi(\beta' - \beta_0)}{180}\left(\frac{\partial\beta_0}{\partial a_0}\right)$$

$$a\Sigma\left(\frac{\partial\beta_0}{\partial a_0}\right)\left(\frac{\partial\beta_0}{\partial\mathrm{B}_0}\right) + b\Sigma\left(\frac{\partial\beta_0}{\partial\mathrm{B}_0}\right)^2 = \Sigma\frac{\pi(\beta' - \beta_0)}{180}\left(\frac{\partial\beta_0}{\partial\mathrm{B}_0}\right),$$

and here we may, in the coefficients, replace δ_0, β_0, by δ', β' wherever convenient. We thus find

$$E_0\left(\frac{\partial \delta_0}{\partial F_0}\right) = -\frac{E_0 \sin^2 \delta'}{\sin^2 i_0 \cos i_0},$$

$$E_0\left(\frac{\partial \delta}{\partial E_0}\right) = \sin \delta' \cos \delta',$$

$$2 \cot^2 \alpha \,.\left(\frac{\partial \beta_0}{\partial A_0}\right) = \cot \beta' \,.\, \cos^2 \beta' \,.\, \frac{\sin i_0 \sin i_1 \cos i_0 \cos i_1 \cos (i_0 + i_1)}{\cos^3 (i_0 - i_1)},$$

$$2 \cot^2 \alpha \,.\left(\frac{\partial \beta_0}{\partial B_0}\right) = \cot \beta' \,.\, \cos^2 \beta' \,.\, \frac{\sin^2 i_0 \sin^2 i_1 \cos i_0 \cos i_1}{\cos^4 (i_0 - i_1)}.$$

This is the method used in most cases, but in the more inaccurate experiments it was easier to find the sums of the squares of the errors for several pairs of values of the constants, and thence, by a kind of interpolation, to find the best values of the constants.

Since the values of E, B are determined independently, the nearness with which they satisfy the relation $E^2 = \frac{\mu_0}{\mu_1} B$ will serve in some measure as a test of the formulæ.

I have for comparison given the deviations from CAUCHY's formulæ, calculated with the given value of ϵ by the experimenter himself. These run roughly parallel with the deviations from the theoretical values, and where there seemed any very great deviation from parallelism, I have recalculated the results of CAUCHY's formula. For instance, JAMIN, for fire-opal, gives incorrect values for $R\perp/R\parallel$ (his J/I). On recalculating from the given values of β, some of his values are found to be the square roots of what they should be.

As an index of the accuracy of agreement, I have given the probable error of a single observation, as calculated by the formula $\pm \cdot 6745 \sqrt{(S/n - 1)}$, where n is the number of observations, S the sum of the squares of the errors.

§ 8. JAMIN (' Annales de Chimie et de Physique,' III^{me} Série, tomes 29 et 31).

Realgar—Air (29, pp. 292 and 295).

$\mu = 2{\cdot}454$; $A = +{\cdot}0254$, $B = {\cdot}06989$, $F = -{\cdot}0022$, $E = +{\cdot}1565$; $\epsilon = +{\cdot}0791$.

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference, CAUCHY.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, CAUCHY.
°		° ′	° ′	′	′				
85	..	..	..	..	..	·979	·972	+·007	+·006
84	..	31 40	32 9	−29	−40	..	..	..	..
83	..	..	..	..	..	·962	·959	+·003	+·001
82	..	27 10	28 0	−50	−54	..	..	..	..
81	..	..	..	..	..	·951	·944	+·007	+·004
80	..	24 10	24 4	+ 6	+12	..	..	..	..
79	..	..	..	..	..	·927	·926	+·001	−·002
78	..	20 0	20 3	− 3	− 3	..	..	..	..
77	..	..	..	..	..	·901	·903	−·002	−·006
76	..	16 20	16 21	− 1	+ 1	..	..	..	..
75	..	..	..	..	..	·879	·870	+·009	+·004
74	..	12 33	12 55	−22	−10	..	..	..	..
73	..	..	..	..	..	·837	·822	+·015	+·009
72	..	9 22	9 52	−30	−21	·800	·788	+·012	+·005
71	..	..	..	..	..	·753	·742	+·011	+·001
70	..	8 30	7 30	+60	+81*	·694	·685	+·009	−·002
69	..	..	..	..	..	·611	·615	−·004	−·012
68	..	6 56	6 23	+33	+53	·523	·529	−·006	−·040
67	..	..	..	..	..	·433	·443	−·010	−·017
66	..	6 46	6 51	− 5	+19	·364	·365	−·001	−·005
65	..	..	..	..	..	·292	·302	−·010	−·012
64	..	8 46	8 30	+15	+36	·251	·253	−·002	−·002
63	..	..	..	..	..	·230	·215	+·015	+·013
62	..	10 33	10 42	− 9	+ 9	·193	·186	+·007	+·008
61	..	..	..	..	..	·170	·163	+·007	+·002
60	..	12 30*	13 17	−47	−12	·154	·145	+·009	+·009
58	..	15 40	15 24	+16	+30	·127	·116	+·011	+·012
56	..	17 33	17 41	− 8	+ 4	·106	·096	+·010	+·011
54	..	19 55	19 55	0	+ 9	·090	·082	+·008	+·001
52	..	21 36	22 0	−24	−14	·075	·070	+·005	+·006
50	..	23 18	23 59	−41	−35	·052	·060	−·008	−·007
48	..	26 30	25 52	+38	−14	·046	·052	−·006	−·006
46	..	..	..	..	..	·043	·046	−·003	−·002
44	..	..	..	..	..	·034	·040	−·006	−·006
42	..	..	..	..	..	·025	·035	−·010	−·009
40	..	32 40	32 19	+21	+28	·024	·031	−·007	−·006
30	..	..	..	..	..	·018	·015	+·003	+·003
Probable error				±20′·82	±22′·96	..	..	±·0054	±·0071

* The observations marked (*) have been recalculated. For the first, JAMIN calculates 8° 9′, giving an error of + 21, which is clearly too small. Recalculation gives 7° 9′, with an error 1° 21′. The second, JAMIN misprints 11° 30′, but he gives calculated 12° 42′, difference − 12′, showing that it should be 12° 30′, which is confirmed by the entry A (azimuth of small axis of vibrational ellipse) = 12°. δ is given in fractions of $\frac{\lambda}{2}$.

Diamond—Air (29, p. 297).

$\mu = 2{\cdot}434$; $A = -{\cdot}0183$, $B = {\cdot}00353$, $F = -{\cdot}00045$, $E = +{\cdot}03577$; $\epsilon = +{\cdot}0180$.

i_0.		β.		ϖ observed.		ϖ calculated.		Difference.	Difference, Cauchy.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, Cauchy.
°	′	°	′	°	′	°	′	′	′				
75	0	66	22	13	30	13	25	+ 5	+13	·962	·969	− ·007	− ·008
74	0	64	7	12	13	11	31	+42	+50	·955	·963	− ·008	− ·009
73	0	58	37	9	46	9	40	+ 6	+ 3	·948	·955	− ·007	− ·008
72	0	52	15	7	44	7	51	− 7	+ 2	·940	·944	− ·004	− ·002
71	0	45	22	5	53	6	5	−12	− 3	·928	·927	+ ·001	·000
70	0	34	52	4	11	4	23	−12	− 3	·897	·896	+ ·001	·000
69	30	31	57	3	45	3	34	+11	+20	·868	·870	− ·002	− ·004
69	0	26	7	2	57	2	48	+ 9	+39	·826	·829	− ·003	− ·003
68	30	18	45	2	3	2	7	− 4	+ 5	·769	·759	+ ·010	+ ·011
68	0	14	0	1	30	1	36	− 6	+ 2	·640	·634	+ ·006	+ ·011
67	75*	13	2	1	23	1	28	− 5	+ 1	·545	·545	000	+ ·007
67	30	12	52	1	22	1	25	− 3	− 1	·437	·439	− ·012	− ·004
67	15	14	37	1	34	1	31	+ 3	+ 2	·363	·362	+ ·001	+ ·009
67	0	16	22	1	45	1	43	+ 2	+ 2	·288	·292	− ·004	+ ·002
66	30	21	35	2	23	2	15	+ 8	+ 3	·202	·201	+ ·001	+ ·005
66	0	27	35	3	9	2	56	+13	− 7	·155	·150	+ ·005	+ ·008
65	0	35	45	4	20	4	24	− 4	− 9	·105	·099	+ ·006	+ ·008
64	0	43	40	5	46	5	51	− 5	+12	·073	·073	·000	+ ·003
63	0	51	45	7	36	7	19	+17	+10	·063	·058	+ ·005	+ ·006
62	0	54	15	8	18	8	45	−27	−36	·047	·047	·000	·000
61	0	59	15	10	1	10	10	− 9	−16	·042	·040	+ ·002	+ ·003
60	0	62	53	11	35	11	32	+ 3	+ 1	·032	·035	− ·003	− ·002
Probable error								± 9′·10	±11′·84	..	..	±·0038	±·0042

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* JAMIN has $i_0 = 67° 55'$, which is a misprint, since JAMIN's own calculation of δ with $i_0 = 67° 55'$ ought to give $\delta = {\cdot}598$, whilst $i_0 = 67° 45'$ gives ·538, the actual number in the table.

Blend—Air (29, p. 296).

$\mu = 2{\cdot}371$; $A = +{\cdot}0275$, $B = {\cdot}01180$, $F = -{\cdot}00060$, $E = +{\cdot}06713$; $\epsilon = +{\cdot}0296$.

i_0.		β.		ϖ observed.		ϖ calculated.		Difference.	Difference, Cauchy.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, Cauchy.
°	′	°	′	°	′	°	′	′	′				
76	0	70	15	16	18	16	2	+16	−39	·955	·956	− ·001	+ ·004
74	0	64	45	12	34	12	17	+17	−14	·936	·939	− ·003	− ·005
72	0	54	15	8	19	8	41	−22	−31	·912	·912	·000	− ·001
70	0	42	0	5	24	5	23	+ 1	− 6	·859	·853	+ ·006	+ ·004
69	0	34	0	4	3	3	57	+ 6	+ 2	·784	·791	− ·007	− ·009
68	0	26	30	3	0	2	51	+ 9	+ 8	·681	·674	+ ·007	+ ·005
67	30	23	37	2	38	2	35	+ 3	+ 6	·594	·585	+ ·009	+ ·008
67	0	22	55	2	33	2	38	− 5	+ 7	·471	·481	− ·010	− ·010
66	30	25	23	2	51	2	47	+ 4	+14	·380	·382	− ·002	·000
66	0	28	45	3	18	3	12	+ 6	+19	·292	·302	− ·010	− ·008
65	30	32	0	3	44	3	44	0	+ 7	·246	·243	+ ·003	+ ·005
65	0	37	25	4	36	4	21	+15	+28	·212	·201	+ ·011	+ ·013
64	0	43	0	5	36	5	40	− 4	+ 6	·151	·147	+ ·004	+ ·006
63	0	50	15	7	12	7	3	+ 9	+21	·124	·115	+ ·009	+ ·011
62	0	54	30	8	23	8	27	− 4	+ 7	·090	·094	− ·004	− ·002
61	0	59	15	10	1	9	49	+12	+23	·075	·079	− ·004	− ·001
60	0	61	45	11	4	11	11	− 7	+ 9	·068	·068	·000	− ·001
Probable error								± 6′·23	±12′·25	..	..	±·0040	±·0046

Flint—Air (29, p. 298).

$\mu = 1{\cdot}714$; $A = -{\cdot}0317$, $B + {\cdot}00260$, $F = +{\cdot}000094$, $E = +{\cdot}0339$; $\epsilon = +{\cdot}0170$.

i_0.		β.		ϖ observed.		ϖ calculated.		Difference.	Difference, Cauchy.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, Cauchy.
°	′	°	′	°	′	°	′	′	′				
65	15	33	15	8	16	8	54	−38	−15	·959	·965	− ·006	− ·006
64	0	29	15	7	5	6	58	+ 7	+ 5	·957	·954	+ ·003	− ·010
63	0	24	30	5	46	5	27	+19	+37	·940	·930	+ ·010	− ·001
62	0	17	52	4	5	3	57	+ 8	+28	·913	·912	+ ·001	+ ·010
61	0	12	15	2	45	2	32	+13	+35	·842	·853	− ·011	+ ·035
60	30	9	10	2	3	1	54	+ 9	+32	·788	·780	+ ·008	+ ·053
60	0	5	31	1	13	1	25	−12	+10	·623	·623	·000	+ ·017
59	30	4	47	1	4	1	16	−12	+ 1	·382	·382	·000	+ ·019
59	0	6	45	1	30	1	32	− 2	+ 0	·223	·222	+ ·001	− ·006
58	30	8	47	1	58	2	5	− 7	−10	·149	·148	+ ·001	+ ·018
58	0	12	14	2	45	2	43	+ 2	− 5	·100	·109	− ·009	+ ·008
57	0	17	42	4	3	3	44	+19	−14	·071	·071	·000	− ·007
56	0	23	15	5	26	5	34	− 8	−20	·052	·052	·000	− ·009
55	0	29	0	7	0	7	1	− 1	− 3	·041	·041	·000	− 002
54	0	33	52	8	27	8	28	− 1	−13	·034	·034	·000	+ ·002
53	0	38	45	10	5	9	54	+11	− 1	·027	·028	− ·001	− ·001
Probable error								± 9′·44	±12′·92	..	..	±·0035	±·0404

Fire-opal—Air (29, p. 279).

$\mu = 1{\cdot}623$; $A = -{\cdot}0040$, $B = {\cdot}00594$, $F = +{\cdot}000063$, $E = +{\cdot}0625$;
$\epsilon =$ not given.

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference, CAUCHY.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, CAUCHY.
° ′	° ′	° ′	° ′	′					
60 0	27 30	3 8	3 4	+ 4	Not calculated.	·843	·810	+·033	Not calculated.
59 45	24 30	2 44	2 44	0		.810	·785	+·025	
59 30	21 30	2 22	2 27	− 5		·734	·753	−·019	
59 15	19 30	2 8	2 11	− 3		·703	·714	−·011	
59 0	18 0	1 57	1 57	0		·666	·664	+·002	
58 45	16 30	1 47	1 47	0		·609	·605	+·004	
58 30	16 0	1 44	1 41	+ 3		·540	·537	+·003	
58 22	15 0	1 37	1 39	− 2		·500	·499	+·001	
58 15	15 15	1 38	1 39	− 1		·455	·465	−·010	
58 0	16 45	1 49	1 43	+ 6		·397	·397	·000	
57 45	17 0	1 50	1 53	− 3		·337	·337	·000	
57 30	19 0	2 4	2 5	− 1		·295	·287	+·008	
57 0	22 45	2 31	2 34	− 3		·220	·215	+·005	
56 30	30 0	3 28	3 16	+12		·163	·169	−·006	
56 0	32 30	3 50	3 56	− 6		·143	·138	+·005	
Probable error				±3′·10	..	..	..	±·0280	

Hyalite—Air (29, p. 281).

$\mu = 1{\cdot}421$; $A = {\cdot}000$, $B = {\cdot}00040$, $F = +{\cdot}00026$, $E = -{\cdot}0150$; $\epsilon = -{\cdot}0074$.

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference, CAUCHY.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, CAUCHY.
° ′	° ′	° ′	° ′	′					
56 0	18 0	1 57	1 51	+ 6	Not calculated.	−·924	−·934	+·010	Not calculated.
55 30	11 30	1 13	1 6	+ 7		−·898	−·885	−·013	
55 15	5 37	0 36	0 45	− 9		−·850	−·822	−·028	
55 0	4 22	0 28	0 28	0		−·641	−·656	+·015	
54 52	4 6	0 26	0 25	+ 1		−·500	−·489	−·011	
54 45	4 15	0 27	0 27	0		−·329	−·345	+·016	
54 30	8 0	0 51	0 43	+ 8		−·177	−·178	+·001	
54 15	10 56	1 10	1 4	+ 6		−·140	−·114	−·028	
53 30	18 30	2 1	2 12	−11		−·092	−·055	−·037	
Probable error				±4′·65	..	..	..	±·0218	

Glass—Air (29, p. 299).

$\mu = 1{\cdot}487$; $A = -{\cdot}0064$, $B = {\cdot}000296$, $F = -{\cdot}00030$, $E = +{\cdot}0154$; $\epsilon = +{\cdot}00752$.

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference, CAUCHY.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, CAUCHY.
61° 0′	51° 55′	7° 38′	7° 44′	−6′	−3′	·981	·984	−·003	−·004
60 0	45 24	6 5	6 12	−7	+36	·978	·980	−·002	−·002
59 0	37 40	4 38	4 38	0	−2	·975*	·973	+·002	+·002
58 0	26 45	3 2	3 5	−3	−48	·958	·959	−·001	−·012
57 30	20 26	2 15	2 18	−3	0	·949	·944	+·005	−·004
57 15	17 2	1 51	1 55	−4	−1	·935	·933	+·002	+·001
57 0	14 56	1 36	1 32	+4	+24	·913	·916	−·003	−·004
56 45	11 17	1 12	1 9	+3	+5	·898	·888	+·010	+·010
56 30	8 7	0 52	0 48	+4	+5	·846	·832	+·014	+·009
56 15	4 37	0 29	0 29	0	+1	·686	·701	−·015	+·005
56 0	3 22	0 21	0 22	−1	−1	·420	·417	+·003	+·024
55 45	5 15	0 33	0 36	−3	−6	·223	·214	+·009	+·064
55 30	8 32	0 54	0 56	−2	−5	·141	·133	+·008	+·014
55 15	11 52	1 16	1 18	−2	+7	·085	·095	−·010	·000
55 0	16 0	1 44	1 41	+3	−1	·058	·073	−·015	−·003
54 30	23 3	2 34	2 27	+7	+4	·046	·050	−·004	−·002
54 0	27 38	3 9	3 13	−4	−8	·036	·038	−·002	−·001
53 30	33 56	4 3	4 1	+2	0	·032	·031	+·001	+·003
Probable error				±2·63	±10·77	..	..	±·0053	±·0119

* JAMIN gives ·985 instead of ·975; but the difference (CAUCHY) ·002 given by JAMIN shows that 8 is a misprint for 7; in any case it does not make much difference.

Fluorspar—Air (29, p. 300).

$\mu = 1{\cdot}441$; $A = +{\cdot}0043$, $B = {\cdot}00080$, $F = +{\cdot}00104$, $E = -{\cdot}0202$; $\epsilon = -{\cdot}00969$.

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference, CAUCHY.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, CAUCHY.
60° 0′	52° 7′	7° 42′	7° 28′	+14′	+13′	−·986	−·978	−·008	−·006
57 30	32 30	3 50	3 36	+14	+15	−·957	−·954	−·003	−·001
57 0	25 52	2 55	2 49	+6	+10	−·943	−·941	−·002	+·001
56 30	18 18	1 59	2 3	−4	−3	−·916	−·917	+·001	+·007
56 0	13 0	1 23	1 19	+4	+6	−·868	−·866	−·002	+·008
55 45	8 10	0 52	0 59	−7	−3	−·819	−·808	−·011	+·003
55 15	6 0	0 38	0 36	+2	+5	−·463	−·467	+·004	+·036
55 0	6 35	0 42	0 43	−1	+5	−·265	−·269	+·004	+·017
54 45	9 15	0 59	1 0	−1	+4	−·175	−·171	−·004	·000
54 30	11 38	1 14	1 20	−6	−3	−·125	−·123	−·002	−·002
54 15	15 15	1 38	1 42	−4	−1	−·099	−·095	−·004	−·005
54 0	20 0	2 11	2 4	+7	+4	−·078	−·077	−·001	−·002
53 30	26 45	3 2	2 50	+12	+14	−·059	−·056	−·003	−·004
53 0	32 0	3 45	3 37	+8	+10	−·051	−·044	−·007	−·009
Probable error				±5·32	±5·56	..	..	±·0034	±·0080

Essence of Lavender—Air (31, p. 173).

$\mu = 1{\cdot}462$; $A = +\,{\cdot}00387$, $B = {\cdot}000027$, $F = -\,{\cdot}00096$, $E = +\,{\cdot}00670$; $\epsilon = +\,{\cdot}00150$.

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference CAUCHY.	$-\frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference CAUCHY.
56° 32′	16° 0′	1° 26′	1° 24′	+ 2′	+ 2′	·967	·962	+ ·005	− ·004
56 20	13 0	1 9	1 5	+ 4	+ 2	·944	·952	− ·008	− ·019
56 14	10 20	0 55	0 56	− 1	− 3	·941	·944	− ·003	− ·017
56 8	8 40	0 46	0 46	0	− 2	·932	·933	− ·001	− ·017
56 2	6 15	0 33	0 37	− 4	− 6	·917	·918	− ·001	− ·021
55 56	4 45	0 25	0 28	− 3	− 5	·922	·893	+ ·029	+ ·004
55 50	4 20	0 23	0 19	+ 4	+ 2	·899	·848	+ ·051	+ ·011
55 44	1 58	0 10	0 11	− 1	0	·758	·752	+ ·006	− ·038
55 38	1 18	0 7	0 6	+ 1	− 1	·501	·528	− ·027	− ·021
55 32	1 40	0 9	0 12	− 3	− 3	·282	·278	+ ·004	+ ·058
55 26	4 20	0 23	0 21	+ 2	+ 4	·187	·164	+ ·023	+ ·064
55 20	5 30	0 29	0 30	− 1	0	·136	·115	+ ·021	+ ·049
55 14	6 22	0 34	0 39	− 5	− 5	·103	·086	+ ·017	+ ·022
55 8	8 50	0 47	0 49	− 2	0	·081	·069	+ ·012	+ ·030
55 2	10 20	0 55	0 58	− 3	− 1	·046	·057	− ·011	+ ·004
54 56	13 10	1 10	1 8	+ 2	+ 4	·046	·048	− ·002	+ ·010
54 50	14 30	1 18	1 16	+ 2	+ 3	·042	·042	·000	+ ·010
54 44	16 50	1 31	1 27	+ 4	+ 4	·034	·038	− ·004	+ ·005
54 38	18 0	1 38	1 36	+ 2	+ 3	·030	·034	− ·004	+ ·005
		Probable error		± 1·60	± 1·67	..	..	±·0102	±·0160

Distilled Water—Air (31, p. 174).

$\mu = 1{\cdot}333$; $A = {\cdot}000$, $B = {\cdot}00016$, $F = -{\cdot}00018$, $E = -{\cdot}0126$; $\epsilon = -{\cdot}00577$.

i_0.		β.		ϖ observed.		ϖ calculated.		Difference.	Difference Cauchy.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference Cauchy.
°	′	°	′	°	′	°	′	′	′				
55	36	39	0	4	3	4	2	+ 1	+11	−·981	−·976	− ·005	− ·005
54	54	23	0	2	8	2	54	−46	−47	−·977	−·967	− ·010	− ·008
54	30	16	0	1	26	2	15	−49	−50	−·957	−·957	·000	+ ·004
54	3	12	30	1	7	1	32	−25	−27	−·937	−·937	·000	+ ·005
53	50	..		..		1	11	..	..	−·917	−·919	+ ·002	+ ·008
53	42	10	37	0	56	0	59	− 3	− 3	−·899	−·902	+ ·003	+ ·010
53	35	8	59	0	48	0	48	0	0	−·884	−·870	− ·014	+ ·004
53	30	7	30	0	40	0	40	0	− 1	−·854	−·857	+ ·003	+ ·006
53	23	5	30	0	29	0	30	− 1	− 2	−·829	−·807	− ·022	− ·010
53	19	5	30	0	29	0	25	+ 4	+ 1	−·727	−·763	+ ·036	+ ·049
53	15	4	30	0	24	0	20	+ 4	− 2	−·703	−·699	− ·004	+ ·009
53	12	3	52	0	20	0	18	+ 2	+ 2	−·623	−·635	+ ·012	+ ·023
53	9	3	0	0	16	0	16	0	− 2	−·554	−·557	+ ·003	+ ·008
53	7	3	0	0	16	0	16	0	0	−·500	−·500	·000	·000
53	6	3	0	0	16	0	16	0	0	−·434	−·472	+ ·038	+ ·036
53	3	3	0	0	16	0	17	− 1	− 2	−·436	−·390	− ·046	− ·054
52	59	3	50	0	20	0	21	− 1	− 1	−·295	−·301	+ ·006	− ·007
52	55	5	4	0	27	0	26	+ 1	+ 2	−·250	−·237	− ·013	− ·026
52	50	6	0	0	32	0	32	0	+ 3	−·153	−·184	+ ·031	+ ·018
52	46	6	4	0	32	0	38	− 6	+ 2	−·134	−·154	+ ·020	+ ·009
52	42	8	55	0	47	0	44	+ 3	+ 8	−·138	−·133	− ·005	− ·009
52	38	10	15	0	54	0	50	+ 4	+ 4	−·106	−·116	+ ·010	+ ·001
52	31	13	15	1	11	1	1	+10	+11	−·087	−·095	+ ·008	·000
52	26	13	50	1	14	1	9	+ 5	+ 5	−·079	−·084	+ ·005	− ·002
52	16	15	54	1	26	1	25	+ 1	+ 2	−·070	−·068	− ·002	− ·008
52	5	19	7	1	44	1	43	+ 1	+ 2	−·058	−·056	− ·002	− ·007
51	45	21	10	1	56	2	15	−19	−10	−·054	−·042	− ·012	− ·015
51	24	24	0	2	14	2	49	−35	−30	−·046	−·034	− ·012	− ·016
50	56	29	0	2	47	3	34	−47	−47*	−·032	−·026	− ·006	− ·008
Probable error								± 12·22	± 12·17	..	..	±·0113	±·0129

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* This is recalculated; Jamin has + 11′, which is certainly wrong.

Ferric Chloride Solution $\frac{1}{6}$—*Air* (31, p. 175).

$\mu = 1{\cdot}372$, $A = +{\cdot}00083$, $B = {\cdot}00068$, $F = +{\cdot}00005$, $E = -{\cdot}0222$, $\epsilon = -{\cdot}01056$.

i_0.		β.		ϖ observed.		ϖ calculated.		Difference.	Difference, CAUCHY.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, CAUCHY.
°	′	°	′	°	′	°	′	′	′				
55	29	26	0	2	27	2	31	− 4	− 8	−·954	−·952	− ·002	− ·019
55	23	25	0	2	20	2	22	− 2	−15	−·938	−·948	+ ·010	− ·001
55	17	24	0	2	14	2	12	+ 2	− 2	−·930	−·922	− ·008	− ·004
55	11	23	0	2	8	2	3	+ 5	+ 1	−·912	−·917	+ ·005	+ ·009
55	5	20	0	1	49	1	54	− 5	− 8	−·930	−·910	− ·020	− ·016
54	59	19	0	1	44	1	45	− 1	− 4	−·911	−·902	− ·009	− ·004
54	53	18	10	1	39	1	36	+ 3	− 1	−·899	−·893	− ·006	− ·001
54	47	15	30	1	23	1	27	− 4	− 6	−·887	−·882	− ·005	·000
54	40	14	0	1	15	1	16	− 1	− 6	−·867	−·866	− ·001	+ ·007
54	35	13	0	1	9	1	9	0	+ 1	−·848	−·851	+ ·003	+ ·010
54	29	10	30	0	56	1	1	− 5	− 3	−·832	−·829	− ·003	+ ·005
54	23	9	0	0	48	0	53	− 5	− 7	−·810	−·802	− ·008	− ·001
54	17	8	30	0	45	0	46	− 1	− 2	−·743	−·764	+ ·021	+ ·030
54	11	8	0	0	42	0	40	+ 2	+ 2	−·706	−·714	+ ·008	+ ·017
54	5	7	0	0	37	0	36	+ 1	+ 2	−·639	−·648	+ ·009	+ ·017
53	59	7	0	0	37	0	33	+ 4	+ 2	−·546	−·564	+ ·018	+ ·029
53	55	7	0	0	37	0	32	+ 5	+ 5	−·500	−·501	+ ·001	·000
53	50	7	0	0	37	0	34	+ 3	+ 5	−·416	−·424	+ ·008	+ ·006
53	44	8	30	0	45	0	39	+ 6	+10	−·361	−·342	− ·019	− ·024
53	38	8	30	0	45	0	44	+ 1	+ 4	−·318	−·277	− ·041	− ·047
53	32	9	0	0	48	0	52	− 4	0	−·252	−·229	− .023	− ·030
53	26	10	0	0	53	0	59	− 6	− 2	−·183	−·193	+ ·010	+ ·004
53	20	12	0	1	4	1	8	− 4	0	−·161	−·166	+ ·005	− ·002
53	14	13	30	1	12	1	16	− 4	0	−·142	−·145	+ ·003	− ·003
53	8	16	0	1	26	1	25	+ 1	+ 5	..	−·128	..	..
53	2	18	0	1	38	1	34	+ 4	+ 8	−·101	−·115	+ ·014	+ ·009
52	56	18	0	1	38	1	43	− 5	− 1	−·099	−·104	+ ·005	·000
52	50	20	0	1	49	1	52	− 3	+ 6	−·092	−·095	+ ·003	− ·001
52	44	22	0	2	1	2	2	− 1	+ 3	−·070	−·087	+ ·017	+ ·013
52	38	24	0	2	14	2	11	+ 3	+ 7	−·060	−·081	+ ·021	+ ·017
52	32	25	0	2	20	2	20	0	− 1	−·047	−·075	+ ·028	+ ·024
52	25	26	0	2	27	2	32	− 5	−38	−·036	−·069	+ ·033	+ ·030
Probable error								±2′·39	±5′·77	..	..	±·0986	±·1159

Glass in Water (31, p. 184).

$\mu = 1{\cdot}115,\ A = {\cdot}000,\ B = {\cdot}0020,\ F = -{\cdot}00107,\ E = +{\cdot}0390,\ \epsilon = +{\cdot}02078.$

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference, CAUCHY.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, CAUCHY.
° ′	° ′	° ′	° ′	′	′				
49 0	21 30	1 58	1 51	+ 7	+ 5	·881	·851	+ ·030	+ ·044
48 48	17 0	1 32	1 33	− 1	0	·823	·818	+ ·005	+ ·029
48 36	14 0	1 15	1 16	− 1	0	·791	·769	+ ·022	+ ·046
48 24	12 0	1 4	1 3	+ 1	+ 2	·705	·686	+ ·019	+ ·040
48 18	11 0	0 58	0 58	0	+ 1	·639	·642	− ·003	+ ·026
48 12	10 15	0 54	0 55	− 1	0	·556	·582	− ·026	+ ·003
48 6	8 0	0 42	0 54	−12	−11*	·493	·514	− ·021	+ ·016
48 0	10 0	0 53	0 54	− 1	− 2	·470	·446	+ ·024	+ ·045
47 54	11 30	1 1	0 59	+ 2	+ 2	·393	·382	+ ·011	+ ·026
47 48	12 30	1 7	1 4	+ 3	+ 4	·292	·327	− ·035	− ·025
47 36	16 30	1 29	1 18	+11	+12	·238	·243	− ·005	− ·001
47 24	18 30	1 41	1 34	+ 7	+ 5	·186	·189	− ·003	+ ·000
47 12	21 30	1 58	1 56	+ 2	+ 5	·182	·152	+ ·030	+ ·028
47 6	22 0	2 1	2 2	− 1	− 5	·158	·139	+ ·019	+ ·021
46 48	25 0	2 20	2 33	−13	−13	·146	·108	+ ·038	+ ·035
Probable error				±4′·19	±4′·23	..	..	±·0155	±·0202

* Recalculated, JAMIN has +6′.

Glass in Ferric Chloride $\frac{1}{6}$ (31, p. 185).

$\mu = 1{\cdot}091;\ A = +{\cdot}0200,\ B = {\cdot}00080,\ F = +{\cdot}000104,\ E = +{\cdot}0278;\ \epsilon = +{\cdot}01355.$

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference, CAUCHY.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, CAUCHY.
° ′	° ′	° ′	° ′	′	′				
48 40	22 15	2 3	2 6	− 3	−13	·910	·914	− ·004	+ ·015
48 25	22 30	2 5	1 40	+25	+18	·898	·894	+ ·004	+ ·004
48 17	17 30	1 35	1 26	+ 9	+ 1	·877	·876	+ ·001	+ ·002
48 2	11 10	0 59	1 0	− 1	+ 4	841	·829	+ ·012	+ ·011
47 50	10 30	0 56	0 45	+11	+ 5	·753	·759	− ·006	− ·010
47 14	7 15	0 39	0 38	+ 1	− 6	·700	·704	− ·004	− ·003
47 39	6 30	0 34	0 35	− 1	− 7	·646	·644	+ ·002	+ ·003
47 32	6 20	0 33	0 34	− 1	− 5	·535	·537	− ·002	+ ·001
47 30	6 30	0 34	0 34	0	0	·500	·505	− ·005	·000
47 27	7 35	0 40	0 36	+ 4	+ 5	·460	·455	+ ·005	+ ·011
47 20	7 55	0 42	0 42	0	+ 3	·351	·350	+ ·001	+ ·008
47 11	9 20	0 49	0 54	− 5	0	·239	·253	− ·014	− ·007
47 4	10 0	0 53	1 5	−12	− 6	·210	·203	+ ·007	+ ·013
46 47	15 0	1 21	1 33	−12	− 5	·141	·133	+ ·008	+ ·012
46 35	20 0	1 49	1 54	− 5	+ 2	·111	·106	+ ·005	− ·003
46 25	24 30	2 17	2 12	+ 5	+22	·091	·090	+ ·001	− ·004
46 16	28 15	2 41	2 28	+13	+20	·082	·079	+ ·003	+ ·005
46 3	34 22	3 25	2 50	+35	+41	064	·068	− ·004	− ·003
45 30	37 0	3 46	3 51	− 5	+13	·027	·049	− ·022	− ·023
Probable error				±8′·28	±9′·50	..	..	±·00645	±·00724

§ 9. KURZ ('Poggendorff, Annalen,' Band 108, p. 588).

Glass in Air.

$\mu = 1{\cdot}5963$; A = ·000, B = ·0085, F = − ·00016, E = + ·074 ; $\epsilon = {\cdot}0365$.

i_0.	ϖ observed.	ϖ calculated.	Difference.	Difference, FRESNEL.	Difference, CAUCHY.	$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, CAUCHY.	Difference, GREEN.
° ′	° ′	° ′	° ′	° ′	° ′					
75 0	23 10	25 42	−2 32	−2 28	−2 28	·991	·981	+·010	+·007	+·001
73 0	21 54	22 52	−0 58	−0 58	−0 58	·991	·977	+·014	+·014	+·011
71 0	18 9	20 1	−1 52	−1 45	−1 45	·986	·972	+·014	+·014	+·011
69 0	15 41	17 6	−1 25	−1 24	−1 17	·985	·966	+·019	+·019	+·016
67 0	12 26	14 8	−1 42	−1 42	−1 33	·975	·957	+·018	+·017	+·014
65 0	9 54	11 8	−1 14	−1 02	−1 22	·961	·944	+·017	+·016	+·013
64 0	9 42	9 38	+0 4	+0 18	+0 4	·953	·935	+·018	+·018	+·014
63 0	7 7	8 8	−1 1	−0 45	−1 1	·933	·922	+·011	+·011	+·007
62 0	6 34	6 38	−0 4	+0 15	−0 4	·916	·903	+·013	+·012	+·008
61 0	4 56	5 11	−0 15	+0 11	−0 15	·882	·875	+·017	+·007	+·003
60 0	4 1	3 47	+0 14	+0 49	+0 14	·856	·825	+·031	+·031	+·028
59 30	3 8	3 9	−0 1	+0 42	−0 1	·818	·784	+·034	+·034	+·031
59 0	2 31	2 35	−0 4	+0 52	−0 5	·732	·725	+·007	+·007	+·005
58 30	2 10	2 10	0	+1 17	0	·603	·641	−·038	−·034	−·035
58 0	1 58	1 58	0	+1 52	0	·503	·526	−·023	−·026	−·027
57 30	2 2	2 4	−0 2	+1 22	−0 3	·393	·406	−·013	·000	·000
57 0	2 13	2 25	−0 12	+0 46	−0 13	·318	·307	+·011	+·016	+·015
56 30	2 54	2 56	−0 2	+0 42	−0 2	·217	·237	−·020	−·016	−·019
56 0	3 2	3 32	−0 30	+0 3	−0 31	·219	·189	+·030	+·034	+·030
55 30	3 38	4 11	−0 33	−0 7	−0 34	·181	·156	+·025	+·028	+·024
55 0	4 21	4 53	−0 32	−0 10	−0 32	·151	·132	+·019	+·022	+·018
54 0	5 49	6 17	−0 28	−0 12	−0 29	·108	·100	+·008	+·010	+·006
53 0	6 56	7 44	−0 48	−0 35	−0 48	·091	·080	+·011	+·013	+·008
52 0	8 51	9 10	−0 19	−0 9	−0 19	·073	·067	+·006	+·008	+·003
51 0	9 50	10 35	−0 45	−0 38	−0 46	·067	·056	+·011	+·012	+·007
50 0	11 39	12 1	−0 22	−0 15	−0 20	·055	·048	+·007	+·007	+·002
48 0	14 9	14 47	−0 38	−0 33	−0 38	·051	·038	+·013	+·014	+·009
46 0	16 28	17 28	−1 0	−0 56	−1 0	·038	·030	+·008	+·008	+·003
44 0	18 54	20 1	−1 7	−1 5	−1 7	·035	·025	+·010	+·011	+·005
42 0	22 16	22 27	−0 11	−0 10	−0 11	·032	·021	+·011	+·012	+·006
40 0	24 51	24 45	+0 6	+0 7	+0 6	·031	·018	+·013	+·014	+·008
38 0	27 0	26 53	+0 7	+0 8	+0 4	..	..	..	..	..
36 0	28 49	28 54	−0 5	−0 4	−0 10	..	..	..	..	..
34 0	29 44	30 51	−1 7	−1 6	−1 7	..	..	..	..	
32 0	33 35	32 36	+0 59	+0 59	+0 56	..	..	..	..	
30 0	34 3	34 13	−0 10	−0 10	−0 11	..	..	..	..	..
Probable error . . .			±34′·50	±37′·98	±34′·46	..	..	±·0056	±·0058	±·0051

§ 10. Quincke ('Poggendorff, Annalen,' Band 128).

Flint-glass in Air (128, p. 367).

$\mu = 1{\cdot}6160$; $A = -{\cdot}0625$, $B = {\cdot}00533$, $F = -{\cdot}00070$, $E = +{\cdot}0562$; $\epsilon = +{\cdot}0290$,
$\mu' = 1{\cdot}609$.

i_0.		β.		ϖ observed.		ϖ calculated.		Difference.		Difference, Cauchy.		δ observed.	δ calculated.	Difference.	Difference, Cauchy.
°	′	°	′	°	′	°	′	°	′	°	′				
70	0	61	41	18	7	18	27	−	20	−	8	·966	·975	− ·009	− ·003
64	0	41	55	9	0	9	32	−	32	−	15	·952	·947	+ ·005	+ ·011
62	0	30	31	5	56	6	28	−	32	−	17	·930	·920	+ ·010	+ ·015
61	0	24	12	4	32	4	57	−	25	−	12	·890	·894	− ·004	+ ·003
60	0	16	44	3	2	3	36	−	34	−	17	·837	·845	− ·008	·000
59	30	16	18	2	57	2	50	+	7	+	18	·786	·802	− ·016	− ·005
59	0	12	35	2	15	2	12	+	3	+	12	·740	·732	+ ·008	+ ·020
58	30	10	1	1	47	1	43	+	4	+	6	·630	·618	+ ·012	+ ·022
58	7	7	15	1	16	1	31	−	15	−	18	·498	·501	− ·003	+ ·006
57	40	8	50	1	34	1	33	+	1	−	9	·359	·365	− ·006	− ·003
57	20	10	25	1	51	1	46	+	5	−	7	·284	·287	− ·003	− ·001
57	0	12	21	2	12	2	5	+	7	−	8	·238	·231	+ ·007	+ ·009
56	30	14	15	2	34	2	40	−	6	−	22	·175	·174	+ ·001	+ ·001
56	0	15	44	2	51	3	20	−	29	−	46	·137	·138	− ·001	− ·001
54	0	29	34	5	44	6	12	−	28	−	45	·069	·073	− ·004	+ ·002
52	0	41	0	8	43	9	6	−	23	−	24	·049	·049	·000	+ ·001
50	0	48	25	11	14	11	58	−	44	−1	0	·034	·036	− ·002	·000
40	0	69	27	25	11	26	11	−1	0	−1	2*	·018	·013	+ ·005	+ ·007
30	0	75	36	34	29	34	37	−0	8	−0	24	·010	·006	+ ·004	+ ·005
Probable error								±16′·92		±21′·19		. .	. .	±·00153	±·00192

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* Recalculated; Quincke has + 16, which is obviously wrong.

I have given δ in fractions of $\frac{1}{2}\lambda$ as in the previous experiments; Quincke himself gives it in fractions of $\frac{1}{4}\lambda$.

μ' is the value of μ Quincke finds it necessary to use for calculating his experiments by Cauchy's formula, in order to obtain any satisfactory agreement with that formula whatever.

Air in Flint-glass (128, p. 368).

$\mu = 0{\cdot}6188$; $\mathrm{A} = +{\cdot}0667$, $\mathrm{B} = {\cdot}0050$, $\mathrm{F} = +{\cdot}00144$, $\mathrm{E} = -{\cdot}0861$; $\epsilon = -{\cdot}0505$.

i_0.	β.	ϖ, observed.	ϖ calculated.	Difference.	Difference, CAUCHY.	δ observed.	δ calculated.	Difference.	Difference, CAUCHY.
° ′	° ′	° ′	° ′	° ′	° ′				
33 58	45 21	10 7	9 26	+ 41	+ 50	−·919	−·942	+ ·023	+ ·018
33 22	34 12	6 50	6 33	+ 17	+ 11	−·905	−·921	+ ·016	+ ·005
32 47	24 41	4 38	4 8	+ 30	+ 26	−·866	−·878	+ ·012	− ·004
32 29	17 30	3 11	2 58	+ 13	+ 6	−·809	−·829	+ ·020	− ·001
32 12	13 2	2 20	2 1	+ 19	+ 7	−·730	−·742	+ ·012	− ·015
31 59	9 5	1 37	1 32	+ 5	− 10	−·614	−·610	− ·004	− ·023
31 51	6 19	1 7	1 26	− 19	− 36	−·498	−·498	·000	− ·005
31 48	6 51	1 13	1 28	− 15	− 30	−·454	−·454	·000	·000
31 36	10 8	1 49	1 47	+ 2	− 10	− 286	−·305	+ ·019	+ ·037
31 18	14 25	2 36	2 40	− 4	− 9	−·243	−·181	− ·062	− ·037
31 0	19 46	3 38	3 42	− 4	− 5	−·121	−·124	+ ·003	+ ·023
30 43	24 47	4 39	4 41	− 2	− 3	−·082	−·094	+ ·012	+ ·030
30 25	31 16	6 7	5 45	+ 22	+ 24	−·064	−·074	+ ·010	+ ·025
29 50	43 56	9 38	7 49	+1 49	+1 56	−·028	−·052	+ ·024	+ ·037
29 16	48 30	11 16	9 41	+1 35	+1 41	−·020	−·039	+ ·019	+ ·030
28 41	55 10	14 13	11 35	+2 38	+2 45	−·015	−·031	+ ·016	+ ·021
Probable error				±38′·64	±41′·24	..	..	±·0288	±·0320

Flint-glass in Water (128, p. 372).

$\mu = 1{\cdot}2096$; $\mathrm{A} = +{\cdot}1667$, $\mathrm{B} = {\cdot}0120$, $\mathrm{F} = +{\cdot}0127$, $\mathrm{E} = +{\cdot}0737$; $\epsilon = +{\cdot}041$, $\mu' = 1{\cdot}2312$.

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference, CAUCHY.	δ observed.	δ calculated.	Difference.	Difference, CAUCHY.
° ′	° ′	° ′	° ′	° ′	° ′				
56 8	43 2	9 21	10 37	−1 16	−2 33	·941	·935	+·006	·000
54 39	35 20	7 8	6 20	+0 48	+0 34	·941	·913	+·028	+ ·012
53 54	29 17	5 39	5 9	+ 30	+ 16	·910	·894	+·016	+ ·037
53 9	21 19	3 56	4 0	− 4	− 17	·899	·863	+·036	+ ·027
52 25	16 35	3 0	2 58	+ 2	+ 10	·824	·812	+·012	+ ·054
52 2	14 16	2 34	2 30	+ 4	− 20	·786	·769	+·017	+ ·029
51 40	10 28	1 52	2 9	− 17	− 33	·709	·710	−·001	+ ·013
51 25	11 48	2 7	2 0	+ 7	+ 3	·662	·655	+·007	+ ·020
51 10	10 55	1 57	1 56	+ 1	+ 2	·555	·587	−·032	+ ·026
50 55	10 17	1 50	1 57	− 7	− 2	·500	·510	−·010	− ·016
50 11	15 8	2 44	2 28	+ 16	+ 31	·299	·303	−·004	− ·011
48 27	19 43	3 37	4 56	−1 19	− 56	·209	·119	+·090	+ 081
47 58	30 6	5 49	5 40	+0 9	+ 32	·127	·099	+·028	+ ·019
45 47	45 10	10 3	9 13	+ 50	+1 5	·071	·057	+·014	+ ·010
Probable error				±25′·15	±35′·10	..	..	±·0211	±·0630

Water in Flint-glass (128, p. 373).

$\mu = 0{\cdot}8267$; $A = -{\cdot}20$, $B = {\cdot}0100$, $F = -{\cdot}0123$, $E = -{\cdot}0751$; $\epsilon = -{\cdot}052$.

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference, Cauchy.	δ observed.	δ calculated.	Difference.	Difference, Cauchy.
° ′	° ′	° ′	° ′	° ′	′				
41 55	40 19	8 31	7 30	+1 1	+ 51	−·831	−·929	+·098	+ ·079
41 18	32 24	6 23	5 58	+0 25	+ 16	−·831	−·911	+·080	+ ·063
40 40	28 15	5 25	4 28	+ 57	+ 51	−·831	−·877	+·046	+ ·026
40 3	19 58	3 40	2 59	+ 41	+ 29	−·786	−·807	+·021	+ ·003
39 45	15 39	2 50	2 23	+ 27	+ 14	−·718	−·737	+·019	+ ·012
39 27	9 40	1 43	1 49	− 6	− 26	−·633	−·620	−·013	− ·011
39 15	7 55	1 24	1 33	− 9	− 35	−·521	−·513	−·008	− ·047
39 2	8 31	1 31	1 26	+ 5	− 24	−·446	−·392	−·054	− ·031
38 50	8 54	1 35	1 29	+ 6	− 26	−·274	−·301	+·027	+ ·121
38 13	16 24	2 58	2 23	+ 35	+ 4	−·111	−·155	+·044	+ ·113
37 36	21 51	4 3	3 43	+ 20	− 8	−·025	−·099	+·074	+ ·120
37 0	26 30	5 2	5 6	− 4	− 32	−·014	−·072	+·058	+ ·091
36 23	32 3	6 18	6 32	− 14	− 18	+·004	−·055	+·059	+ ·092
35 47	38 3	7 51	7 56	− 5	+ 41	+·004	−·045	+·049	+ ·071
Probable error				±20′·28	±20′·90	..	..	±·0383	±·0516

Crown-glass in Air (128, p. 375).

$\mu = 1{\cdot}5149$; $A = -{\cdot}0300$, $B = {\cdot}00040$, $F = -{\cdot}00283$, $E = +{\cdot}0113$; $\epsilon = +{\cdot}00502$,
$\mu' = 1{\cdot}510$.

i_0.	β.	ϖ observed.	ϖ calculated.	Difference.	Difference, Cauchy.	δ observed.	δ calculated.	Difference.	Difference, Cauchy.
° ′	° ′	° ′	° ′	′	′				
80 0	74 50	32 59	33 26	−27	−26	·943	·998	−·055	−·054
60 0	26 51	5 6	5 35	−29	−24	·897	·981	−·084	−·087
58 0	14 36	2 38	2 29	+ 9	+15	·888	·961	−·073	−·076
57 30	9 34	1 42	1 42	0	+ 5	·877	·944	−·067	−·070
57 0	6 9	1 5	0 58	+ 7	+17	·870	·896	−·026	−·028
56 40	3 38	0 38	0 31	+ 7	+16	·777	·772	+·005	+·010
56 30	2 11	0 22	0 23	− 1	+ 7	·568	·579	−·011	+·024
56 20	2 42	0 29	0 24	+ 5	+ 8	·321	·315	+·006	+·055
56 10	2 30	0 26	0 33	− 7	− 5	·162	·180	−·018	+·007
56 0	5 2	0 53	0 46	+ 7	+ 5	·063	·121	−·058	−·041
55 30	8 33	1 31	1 30	+ 1	− 2	·049	·059	−·010	−·003
55 0	11 9	1 59	2 16	−17	−20	·035	·038	−·003	+·001
54 0	18 56	3 28	3 35	− 7	−20*	·028	·022	+·006	+·008
52 0	31 58	6 17	6 51	−34	−40	·025	·012	+·013	+·014
50 0	44 48	9 56	9 52	+ 4	− 1	·003	·007	−·004	−·011
Probable error				±10′·39	±12′·13	..	..	±·0269	±·0284

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* Recalculated.

Air in Crown-glass (128, p. 376).

$\mu = 0{\cdot}6601$; $A = -{\cdot}0667$, $B = {\cdot}00111$, $F = -{\cdot}00583$, $E = -{\cdot}0250$; $\epsilon = -{\cdot}0173$.

i_0.		β.		ϖ observed.		ϖ calculated.		Difference.		Difference, CAUCHY.		$\delta - \frac{\lambda}{2}$ observed.	δ calculated.	Difference.	Difference, CAUCHY.
°	′	°	′	°	′	°	′	°	′	°	′				
34	31	28	43	5	31	4	31	+1	0	+1	4	−·936	−·963	+·027	+·019
34	12	17	23	3	10	3	20	−0	10	−0	5	−·948	−·954	+·006	−·008
33	52	14	6	2	32	2	8	+0	24	+0	29	−·930	−·934	+·004	−·026
33	33	7	18	1	18	1	3	+0	15	+0	19	−·922	−·850	−·072	−·130
33	27	4	14	0	45	0	45	0	0	+0	2	−·760	−·760	·000	−·066
33	20	2	28	0	26	0	32	−0	6	−0	10	−·489	−·502	+·013	+·010
33	14	3	21	0	35	0	34	+0	1	−0	6	−·281	−·263	−·018	+·044
32	54	7	41	1	22	1	29	−0	7	−0	16	−·101	−·077	−·024	+·015
32	35	12	20	2	12	2	32	−0	20	−0	30	−·077	−·044	−·033	−·009
31	57	26	1	4	55	4	39	+0	16	+0	5	−·064	−·024	−·040	−·028
27	34	62	5	18	24	17	27	+0	57	+0	44*	−·064	−·004	−·060	−·056
25	34	67	4	22	37	22	13	+0	24	+0	8*	−·077	−·003	− 074	−·072
Probable error								±13′·69		±13′·37		. .	. .	±·0277	±·0371

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* Recalculated.

§ 11. *Discussion of the Preceding Experiments.*

In the following Table of Constants, as above determined, I, I′ are respectively the angles of incidence for which δ is $\frac{1}{4}\lambda$, and $R\perp/R\|$, or ϖ is a minimum. They are given as calculated from the given values of the constants. According to Fresnel $\tan I = \tan I' = \mu_1/\mu_0$.

Media.	μ_1.	μ_0.	$\mu = \frac{\mu_1}{\mu_0}$.	A.	B.	μE^2.	E.	F.	tan I.	tan I′.	ϵ.
Realgar—Air	2·454	1·000	2·454	+·0254	·06989	·0612	+·1565	—·0022	2·435	2·435	+·0791
Diamond—Air	2·434	1·000	2·434	—·0183	·00353	·00312	+·0358	—·00045	2·430	2·420	+·0180
Blend—Air	2·371	1·000	2·371	+·0275	·01180	·0107	+·0671	—·00060	2·366	2·400	+·0296
Flint—Air	1·714	1·000	1·714	—·0317	·00260	·00197	+·0339	+·00009	1·714	1·703	+·0170
Fire-opal—Air	1·623	1·000	1·623	—·0040	·00594	·00634	+·0625	+·00006	1·623	1·621	..
Glass (I.)—Air	1·487	1·000	1·487	—·0064	·000296	·00034	—·0154	+·00030	1·486	1·484	+·00752
Fluorspar—Air	1·441	1·000	1·441	+·0043	·00080	·00060	—·0202	+·00104	1·443	1·439	—·00969
Hyalite—Air	1·421	1·000	1·421	·0000	·00040	·00032	—·0150	+·00026	1·422	1·420	—·0074
Essence of lavender—Air	1·462	1·000	1·462	+·00387	·000027	·000065	+·0067	—·00096	1·462	1·459	+·0015
Water—Air	1·333	1·000	1·333	·000	·00016	·00021	—·0126	—·00018	1·333	1·332	—·00577
Ferric chloride 1 : 6—Air	1·372	1·000	1·372	+·00083	·00068	·00068	—·0222	+·00005	1·372	1·373	—·01056
Glass (I.)—Water	1·487	1·333	1·115	·000	·0020	·00170	+·0390	—·00107	1·113	1·114	+·02078
Glass (I.)—Ferric chloride, $\frac{1}{3}$	1·487	1·372	1·091	+·020	·00080	·00084	+·0278	+·00010	1·091	1·092	+·01355
Glass (II.)—Air	1·5963	1·000	1·5963	·000	·0085	·0087	+·0740	—·00016	1·594	1·592	+·0365
Flint-glass—Air	1·6160	1·000	1·6160	—·0625	·00533	·00511	+·0562	—·00070	1·607	1·599	+·0290
Air—Flint-glass	1·000	1·6160	0·6188	+·0667	·0050	·00417	—·0861	+·00144	0·621	0·622	—·0505
Flint-glass—Water	1·6160	1·336	1·2096	+·1667	·0120	·0067	+·0737	+·0127	1·234	1·233	+·041
Water—Flint-glass	1·336	1·6160	0·8267	—·200	·0100	·0050	—·0751	—·0123	0·812	0·810	—·052
Crown-glass—Air	1·5149	1·000	1·5149	—·0300	·00040	·0002	+·0113	—·00283	1·505	1·507	+·00502
Air—Crown-glass	1·000	1·5149	0·6601	—·0667	·00111	·00042	—·0250	—·00583	0·649	0·655	—·0173

The glass (I.) is Jamin's, (II.) is Kurz'; the others are Quincke's.

The expressions for A, B, D, E are given on p. 856. Considering the values of A, B — C, G, J, given on the same page, we see that A, G do not change when the two media on either side of the variable layer are interchanged. B — C merely changes sign, and J does the same—this is evident from the physical meaning of J; we have to take an element P and another element Q, form the expression $\frac{\mu_P^2}{\mu_Q^2} - \frac{\mu_Q^2}{{}_P^2}$, multiply by the product of the elements, and then sum, first, for all elements Q on that side of P from which the light comes, and, lastly, for all the elements P of the film; calling the result J, J′, according as the light comes from one or the other side of the layer, clearly in forming the sum J + J′ we must sum $\frac{\mu_P^2}{\mu_Q^2} - \frac{\mu_Q^2}{\mu_P^2}$ for all elements Q and for all elements P, hence J + J′ vanishes, since the two elements of the integral for any two points P, Q destroy each other.

An inspection of the values of A, B shows that they should have the same values, whether the light comes from one side or the other—provided, of course, the layer remains the same.

As for D, E — D becomes $+ 2\mu_1 \frac{A - \mu_0^2}{\mu_1^2 - \mu_0^2} \cdot \frac{2\pi d}{\lambda}$, E becomes

$$+ 2\mu_1 \frac{A - \mu_0^2 - \mu_1^2 + G\mu_0^2\mu_1^2}{\mu_1^2 - \mu_0^2} \cdot \frac{2\pi d}{\lambda}.$$

It has already been stated that B, E are not independent constants; by theory we have $B = \frac{\mu_1}{\mu_0} \cdot E^2$.

A comparison of the values of B and μE^2, as given in the table, p. 876, shows that this last condition is, with few exceptions, very nearly fulfilled. The chief exception is in the case of essence of lavender, where B is ·000027, whilst μE^2 is ·000065, but this is sufficiently accounted for by the smallness of B, and the consequent smallness of ϖ and β, which makes a small error in the determination of β important relatively to the magnitude of B. The large differences in the last four pairs in the table on p. 876 may be due to terms of the third order in E, but these sets of experiments are not very accurate, the contact of liquids and solids being irregular in character. Of the two constants, E is determinable with much the greater accuracy, since the variations from FRESNEL'S formulæ, which are given by all the constants = zero, are much greater for the phases than for the intensities, but it is not easy to say what weight should be attached to each determination. I myself should prefer to rely solely on the value of E, and thence calculate B; this will not very much alter the values of $\tan^2 \varpi$, which are chiefly determined by the values of $\frac{\cos^2(i_0 + i_1)}{\cos^2(i_0 - i_1)}$. This is confirmed by the experiments of KURZ on flint-glass in air (p. 871), where FRESNEL'S formula is seen to give nearly as good a representation of the intensities as the theoretical formula and that of CAUCHY.

The only experiments bearing on the relations between the constants for reflection from either of several pairs of media are those of QUINCKE for flint-glass—air, flint-glass—water, and crown-glass—air. These experiments are very irregular, as shown by the very large "probable errors" occurring in all except the first. QUINCKE himself admits that he could not attain to the accuracy of JAMIN and even of KURZ, and, as already stated, in order to make CAUCHY's formulæ fit at all, he has to use a different value of μ from that which is determined in the ordinary way. For instance, for flint-glass—air he uses 1·609 in place of 1·6160, for flint-glass—water 1·2312 in place of 1·2096, and for crown-glass—air 1·510 instead of 1·515. He gives several other sets of experiments in addition to these, but they consist of few observations and are very much more unreliable still.

As stated above (p. 877) A, B should be the same for the two sets of experiments on each pair. In the case of A this is certainly not true. For flint-glass—air and flint-glass—water they are of opposite sign. The determination of A depends almost entirely on the extreme terms of the series of observations, for it is multiplied by $\cos(i_0 + i_1)$, which is very small for the middle terms. Now the extreme observations in these experiments of QUINCKE's show very large errors indeed, in some cases of more than a degree in ϖ, and are not to be much relied upon. The entire extinction of A would not make a difference of more than a few minutes, and if we decide to retain it, little stress can be laid on its not satisfying the theoretical conditions.*

The case of B and E is much more important, as the deviations from FRESNEL's formulæ depend on them to a first approximation.

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* *On the Accuracy with which the Constants are determined.*

The expressions on p. 860 give

$$\frac{\partial\varpi}{\partial A} = \frac{\sin i_0 \sin i_1 \cos i_0 \cos i_1}{2\cos^2(i_0 - i_1)}\cos^2\varpi, \qquad \frac{\partial\varpi}{\partial B} = \frac{\sin^2 i_0 \sin^2 i_1 \cos i_0 \cos i_1}{2\cos^4(i_0 - i_1)}\cos^2\varpi \, . \cot\varpi,$$

$$\frac{\partial\delta}{\partial F} = -\frac{\sin^2\delta}{\sin^2 i_0 \cos i_0}, \qquad \frac{\partial\delta}{\partial E} = \frac{\sin 2\delta}{2E}.$$

Let us consider the effect of small errors of 10′ in ϖ, and of $\frac{1}{2}/\lambda$ in δ, say for a glass such as that used by KURZ (p. 871) at an angle of incidence of 60°. For this angle ϖ is about 4°, δ is about $\frac{10}{24}\lambda$ or 150°. We find

$$\frac{\partial\varpi}{\partial A} = \cdot 125, \quad \frac{\partial\varpi}{\partial B} = \cdot 915, \quad \frac{\partial\delta}{\partial F} = -\cdot 667, \quad \frac{\partial\delta}{\partial E} = -5\cdot 85,$$

$d\varpi$, $d\delta$ being measured in radians.

The circular measure of $d\varpi$ — 10′ — is ·0029, that of $d\delta$ — ·010 of $\frac{\lambda}{2}$ or 1° 8′ — is ·0200.

Thus,

$$d\varpi = 10' \text{ could be produced by } dA = \cdot 023, \text{ or by } dB = \cdot 0032,$$

and

$$d\delta = \tfrac{1}{100} \text{ of } \tfrac{1}{2}\lambda \text{ could be produced by } dF = -\cdot 030, \text{ or by } dE = -\cdot 0035.$$

Thus,

The values of B, E for the three pairs, are

Reflection from flint-glass in air	B = ·00533,	E = + ·0562,	μ_1E = + ·0908
,, ,, air in flint-glass	·0050,	− ·0861,	− ·0861
,, ,, flint-glass in water	·0120,	+ ·0737,	+ ·1186
,, ,, water in flint-glass	·0100,	− ·0751,	− ·1004
,, ,, crown-glass in air	·00040,	+ ·0113,	+ ·0171
,, ,, air in crown-glass	·00111	− ·0250,	− ·0250

Here μ_1, of course, is the absolute refractive index of the second medium. It will be seen that the relations B = B′ and $\mu_1\text{E} = -(\mu_1\text{E})'$ are satisfied with fair accuracy for the first two pairs, whilst for the third they are only of the same order of magnitude.

According to CAUCHY and JAMIN the ellipticities ϵ, ϵ' in such cases ought to satisfy the relation $-\epsilon'/\epsilon = \mu_1/\mu'_1$. These ratios $-\epsilon'/\epsilon$ are 1·741, 1·244, and 3·446 instead of 1·616, 1·210, and 1·515, and the agreement is less than for our theoretical formula.

Of course it has throughout been assumed that the nature of the film of transition is the same in both sets of experiments. The outstanding difference from agreement may possibly be due to a change in the film. DRUDE (' WIED. Ann.,' 38, p. 35) by observations on cleavage faces of calc-spar has shown that there is in that case a gradual change in the elliptic polarization during exposure, so that part of the effect at least must be ascribed to condensed air or dust, and it is quite possible that such a layer would be affected by atmospheric conditions.

Without some assumption as to the law of variation of refractive index in the layer, there is no relation between the constants for sets of media other than those given above. Theoretically CAUCHY's constant ϵ_{12} for reflection from medium (1) in medium (2) should satisfy the equation $\frac{\epsilon_{12}}{\mu_1} + \frac{\epsilon_{23}}{\mu_2} + \frac{\epsilon_{31}}{\mu_3} = 0$, but this is very

A is determined with an accuracy only about $\frac{1}{7}$ that of B,
and
F ,, ,, ,, ,, $\frac{1}{9}$,, E.

In the experiments of KURZ just quoted, the "probable error" of ϖ is about 35′, that of δ about ·0060 of $\frac{1}{2}\lambda$. Hence, in this case the accuracy for B is only about ·0021/·0112, or $\frac{2}{11}$ that of E.

But in most cases the disparity is not so great.

The last constant F is of the second order in $\frac{2\pi d}{\lambda}$, and in most cases is only from $\frac{1}{1000}$th to $\frac{1}{50}$th of E; the exceptions being fluorspar—air $\frac{1}{20}$, essence of lavender—air $\frac{1}{7}$, and flint glass—water and crown-glass—air, $\frac{1}{4}$—$\frac{1}{7}$. In the case of essence of lavender E is very small. The last four pairs involve the most inaccurate measurements of all those considered. The effect of F is to make the polarizing angle differ from BREWSTER's angle by an amount $2\frac{\mu_1}{\mu_0}$F radius or $\frac{115\mu_1}{\mu_0}$F degrees; this for realgar is about $\frac{1}{2}$°, for flint-glass—water about $1\frac{3}{4}$°, for most other substances $\frac{1}{10}$° or so. As it is unlikely that the polarizing angle can be determined with an accuracy of 1 minute of arc, it is clear that F is known only roughly.

far from being the case. In so far as no such relation exists for our theory, it has the advantage over CAUCHY'S.

Let us now consider the values of D, E in greater detail; by p. 836 the values of these constants are, leaving out of account terms in $(2\pi d/\lambda)^3$,

$$D = 2\mu_0 \cdot \frac{\mu_1^2 - A}{\mu_1^2 - \mu_0^2} \cdot \frac{2\pi d}{\lambda}, \qquad E = -2\mu_0 \frac{A - \mu_0^2 - \mu_1^2 + G\mu_0^2\mu_1^2}{u_1^2 - \mu_0^2} \cdot \frac{2\pi d}{\lambda},$$

where A, G are the mean values of μ^2, $\frac{1}{\mu^2}$ for the variable layer respectively.

We have

$$A - \mu_0^2 - \mu_1^2 + G\mu_0^2\mu_1^2 = \frac{1}{d} \cdot \int_0^d \left(\mu - \frac{\mu_0^2}{\mu}\right)\left(\mu - \frac{\mu_1^2}{\mu}\right) dx.$$

$\left(\mu - \frac{\mu_0^2}{\mu}\right)\left(\mu - \frac{\mu_1^2}{\mu}\right)$ vanishes, when $\mu = \mu_1$ or μ_0 (of course values of $\mu < +1$ do not occur), it has an algebraic minimum $-(\mu_1 - \mu_0)^2$ for the value of $\mu = \sqrt{(\mu_1\mu_0)}$, is negative for values of μ between μ_0 and μ_1, is positive for values either less or greater than both μ_0 and μ_1.

Hence if μ for the variable layer lie between μ_0 and μ_1, $A - \mu_0^2 - \mu_1^2 + G\mu_0^2\mu_1^2$ is certainly negative, if outside those limits, certainly positive. In any other case nothing can be said *à priori* as to its sign, unless indeed the law of variation of μ in the variable layer be given.

If then μ lie between μ_0 and μ_1, E will be positive or negative—and the same will be the character of the reflection in JAMIN'S sense—according as the first medium is the more refractive or the less. And the reverse holds when μ is outside the given limits.

Now JAMIN'S and the other experiments show that the reflection is in most cases (but not in all) positive or negative according as μ_1/μ_0 is greater or less than 1·46. In these cases, we are at liberty to suppose that for positive reflection, that is, when $\mu_1/\mu_0 > 1{\cdot}46$, μ for the film $< \mu_1$, and that for negative reflection, when $\mu_1/\mu_0 < 1{\cdot}46$ (but > 1) μ for the film $> \mu_1$. This shows that when the second medium is air (as is tacitly assumed by JAMIN, otherwise the critical value might be different), the refractive index of the films is, for some parts at least, $> 1{\cdot}46$, and less than 2·5 or so, or perhaps we ought more properly to say that the average refractive index is between those limits. KUNDT has shown that the refractive index of colcothar, or red oxide of iron, which is a common polishing material, is about 2·66; that of chalk, I suppose, would be of the same order of magnitude as for calc-spar and arragonite, that is, about 1·5—1·6. A glass surface, with lumps of such polishing material embedded in it, might be expected to behave as if coated with a film of average refractive index between 1·5 and 2·5, and thus certainly give positive reflection. Of course it has not been proved that μ for every part of the film must lie between the above limits, this is only a sufficient, not a necessary condition.

There is another point to be considered, the magnitude of $2\pi d/\lambda$. It was shown (p. 857) that if $2\pi d/\lambda$ be less than the reciprocal of the greatest value of μ for the film, the expressions found will be convergent. It follows that $| \text{D} | < \dfrac{2\mu_0}{\mu \text{ max.}} \cdot \dfrac{\mu_1^2 \backsim \text{A}}{\mu_1^2 \backsim \mu_0^2}$,

$$| \text{E} | < \frac{2\mu_0}{\mu \text{ max.}} \cdot \frac{\text{A} + \mu_0^2\mu_1^2\text{G} \backsim (\mu_1^2 + \mu_0^2)}{\mu_1^2 \backsim \mu_0^2} .$$

These are the limits which the absolute values of D, E must not exceed. Consider, for instance, the case of water-flint-glass, for which $\mu_1 = 1{\cdot}336$, $\mu_0 = 1{\cdot}616$, $| \text{E} | = {\cdot}075$.

If μ lies between μ_1 and μ_0, then μ max. $= \mu_0 = 1{\cdot}616$; the greatest numerical value of $\text{A} - \mu_0^2 - \mu_1^2 + \text{G}\mu_0^2\mu_1^2$ occurs for $\mu = \sqrt{(\mu_0\mu_1)}$ throughout, and is $(\mu_1 - \mu_0)^2$ (see p. 880) or $({\cdot}28)^2$. The greatest value of $\mu_1^2 \backsim \text{A}$ is given by $\mu = \mu_0$ and is therefore $\mu_0^2 - \mu_1^2$. Hence we must have

$$| \text{D} | < \frac{2 \times 1{\cdot}616}{1{\cdot}616} < 2 ; \qquad | \text{E} | < \frac{2 \times 1{\cdot}616}{1{\cdot}616} \cdot \frac{({\cdot}28)^2}{(1{\cdot}616)^2 - (1{\cdot}336)^2} < 2 \times \frac{{\cdot}28}{2{\cdot}95} < {\cdot}190.$$

If $\mu > \mu_0$, μ max, $= 2{\cdot}67$ (about the greatest value of μ known for a transparent substance), the maximum of $\mu_1^2 \backsim \text{A}$ is $(2{\cdot}67)^2 - (1{\cdot}336)^2$, that of $\text{A} - \mu_0^2 - \mu_1^2 + \text{G}\mu_0^2\mu_1^2$ is $\left\{2{\cdot}67 - \dfrac{(1{\cdot}616)^2}{2{\cdot}67}\right\} \left\{2{\cdot}67 - \dfrac{(1{\cdot}336)^2}{2{\cdot}67}\right\}$. Hence

$$| \text{D} | < \frac{2 \times 1{\cdot}616}{2{\cdot}67} \cdot \frac{(2{\cdot}67)^2 - (1{\cdot}336)^2}{(1{\cdot}616)^2 - (1{\cdot}336)^2} < \frac{3{\cdot}232}{2{\cdot}67} \cdot \frac{5{\cdot}35}{0{\cdot}83} < 7{\cdot}74 ;$$

$$| \text{E} | < \frac{2 \times 1{\cdot}616}{2{\cdot}67} \cdot \frac{\{(2{\cdot}67)^2 - (1{\cdot}616)^2\}\, \{(2{\cdot}67)^2 - (1{\cdot}336)^2\}}{(2{\cdot}67)^2 . \{(1{\cdot}616)^2 - (1{\cdot}336)^2\}} < \frac{3{\cdot}232}{2{\cdot}67} \cdot \frac{4{\cdot}52 \times 5{\cdot}35}{7{\cdot}13 \times 0{\cdot}83} < 5{\cdot}00.$$

These expressions show that it is possible (or at least probable) to satisfy the conditions for convergence by conceivable values of μ for the layer. And since these upper limits for $| \text{D} |$, $| \text{E} |$ are much wider on the second supposition, and rather close to the actual values of $| \text{D} |$,* $| \text{E} |$ on the first supposition, there is a very distinct presumption in favour of the second, namely, that the average value of μ for the variable layer is greater than 1·616 (and less than 2·67). QUINCKE does not state whether his reflecting surfaces were polished with ferric oxide or not, but this is a common enough material. Emery also has a higher refractive index than 1·616, so also diamond dust, and some one of these would perhaps be used, chalk or silica being hardly hard enough for the purpose.

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* The retardation of phase of light polarized in the plane of incidence is $\tan^{-1}$ (D cos i_0). WERNICKE finds that this retardation is at most a few thousandths of a wave-length, so that D is probably less than ·01, and quite incapable of reasonably accurate measurement.

It has already been pointed out (p. 858) that the above supposition would give a value of d less than that for the red of the first order of thin plates, so that no colours of thin plates are to be expected. The constants A, . . . , of course vary with the colour, but their effect, in any case, would not equal that due to variation of i_1, and therefore of $\cos(i_0 + i_1)$ and $\cos(i_0 - i_1)$.

[Owing to the secondary importance of the constants A, D, F, and the impossibility of measuring them accurately, it will be necessary to take account only of E in discussing the limitations to which any law of variation of the refractive index μ in the variable layer is subject. In any particular case, the law must be such as to make μ^2 continuous in value throughout the layer and equal to μ_0^2 and μ_1^2 at the two boundaries; and to give to E its experimental value by a sufficiently small choice of the thickness of d of the layer to ensure convergence of the series for the displacements. Besides, μ^2 must nowhere be less than 1, and nowhere greater than about 10, this last representing the greatest value of μ^2 known to exist for a transparent substance.

The law of variation must involve at least two disposable constants in addition to d.

If the law is to be a general law, so as to include every known case, then it must be capable of making E positive and negative, corresponding to positive and negative reflection. That is, μ^2 must be capable of maxima or minima. For example, the law of variation discussed by Lord RAYLEIGH ('Proc. Lond. Math. Soc.,' XI., p. 51) will not satisfy this condition. In this case, we have $\mu = \frac{\mu_0}{1 - \frac{x}{a}}$, x being the distance from the first face of the variable layer; this gives $\frac{d}{a} = \frac{\mu_1 - \mu_0}{\mu_1}$, $\mathrm{E} = \frac{4}{3}\mu_0 \frac{\mu_1 - \mu_0}{\mu_1 + \mu_0} \cdot \frac{2\pi d}{\lambda}$, which is always positive when the second medium is the more refractive. Hence, Lord RAYLEIGH's law will only explain positive reflection.

If the first medium have a refractive index 1, then μ^2 must have a maximum to give negative reflection.

If the second medium have a refractive index equal to the upper limit, that is 3 or so, then μ^2 must have a minimum in order to give negative reflection.

In addition the general law must make E vanish, that is, $\mu_1^2 + \mu_0^2 = \mathrm{A} + \mu_0^2\mu_1^2\mathrm{G}$, when $\mu_0 = 1$, $\mu_1 = 1{\cdot}46$ or so, in order to explain JAMIN's results.

It follows from GLADSTONE and DALE's experiments, and others of the same kind, that the law of variation of μ^2 may be of the same form as that of the density. The effect of capillary forces will be to make the density vary near the surface of a liquid, possibly also of a solid. A somewhat problematical investigation of the law of variation of the density in the transition film between a liquid and its vapour is given by J. CLERK MAXWELL, in his article on Capillary Action, in the 'Encyclopædia Britannica' (9th Ed.), which gives the density of the variable portion an exponential function of the distance from the surface. If such a law represent the actual circumstances, then negative reflection must be ascribed to adventitious films of dust or condensed gases. It is worth noting that water, which gives strong negative reflection when its surface is covered with grease to even a small amount, when perfectly clean shows hardly any elliptic polarization by reflection (Lord RAYLEIGH,* RÖNTGEN†). Again, various specimens of glass, whose surfaces have been repeatedly cleaned by a method due to WERNICKE, of removing the polishing material by an adhesive coating of gelatine, show much greater positive reflection than when polished with oxide of iron or oxide of tin (WERNICKE, 'WIED. Ann,' 30, p. 402, and K. E. F. SCHMIDT, 'WIED. Ann.,' 51, p. 417, and 52, p. 75). It is clear that the effect of a highly refractive surface film, either of grease or of polishing material, is to produce negative reflection which is superimposed on the effect due to a gradual transition between the ether inside a body and that outside. This latter we should expect to depend on the same causes that produce dispersion and absorption (SCHMIDT, *loc. cit.*, p. 89). Dispersion is taken account of through the refractive index. The absorption effect can be conveniently treated by supposing the refractive index everywhere complex of the form $\mu(1 + \iota\epsilon)$. The distance in which, by absorption, the amplitude is reduced to $1/e$ of its original value is $\lambda/2\pi\mu\epsilon$. In a metal this distance may be as little as $\frac{1}{1000}$th of a wave-length, in a very transparent substance such as glass it may be as much as 100,000 wave-lengths. These values would give $\epsilon =$ about 100, about $\frac{1}{1,000,000}$ respectively. In the one case ϵ is large, in the other it is very small compared with $2\pi d/\lambda$, which must be less than $\frac{1}{10}$. In considering such substances as glass, we may take account of quantities of order ϵ, but may neglect all of higher order.

The effect of absorption on the values of A, B, F, is of order $\epsilon \,.\, 2\pi d/\lambda$, and may be neglected. The effect on D, E is of order ϵ.

The new value of E is

$$- 2\mu_0 \frac{A - \mu_0^2 - \mu_1^2 + G\,\mu_0^2\mu_1^2}{\mu_1^2 - \mu_0^2} \frac{2\pi d}{\lambda} + 2\,\frac{\mu_0}{\mu_1}\,(\epsilon_1 - \epsilon_0),$$

where ϵ_0, ϵ_1 are the values of ϵ for the first and second medium respectively. No term of order ϵ due to the film itself occurs. Hence any small degree of opacity in the film changes the retardation of phase, if at all, by a whole number of wave-lengths. WERNICKE ('WIED. Ann.,' 51, p. 449) finds that whilst there is normally a retardation of phase of $\frac{1}{4}$ wave-length, when light is reflected perpendicularly in glass from an opaque layer of silver closely adhering to the glass, yet the presence of a trace of dust or air between glass and silver suffices to produce instead an acceleration of phase $\frac{3}{4}$ wave-length.

If the more refractive medium be also the more absorptive, as is generally the case, absorption increases positive reflection (since $\epsilon_1 > \epsilon_0$); and of two substances having the same refractive index, the more absorptive will show greater positive (or less negative) reflection, when they are placed in contact with the same third substance. This agrees with the conclusion arrived at by SCHMIDT (*loc. cit.*), from his experiments on various crown and flint-glasses.—July 20.]

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* 'Phil. Mag.' (5), 33, p. 1.

† 'WIED. Ann.,' 46, p. 152

The above experiments are sufficient to test the accuracy of the theory, which merely assumes the existence of a film of transition, without entering into the question of its origin and constancy; whether it be due to a surface property of the medium—a kind of capillary effect—or merely to an adventitious film of dust or of polishing powder, is of no consequence as far as the theory is concerned, its existence is the crucial hypothesis, and of that existence there can be no doubt. The theory does however point to the idea that the film may be, in part at least, of adventitious origin.

This is confirmed by the experiments of DRUDE already mentioned, and those of Lord RAYLEIGH on the reflection from pure water surfaces ('B. A. Repts.,' 1891, p. 563), who finds that perfectly clean water reflects only $\frac{1}{1000}$ of perpendicularly polarized light found by JAMIN, so that its ellipticity is only about $\frac{1}{32}$ of JAMIN's value. The darkness of the band observed in the analyser at the polarizing angle was disturbed by a small trace of olive oil applied to the surface and producing a thin film.

§ 12. *Conclusion.*

We may sum up the results of the preceding discussion as follows :—

(1.) A rigid elastic solid theory, proceeding on the assumption that the velocity of the pressural-wave is much greater than that of the light-wave, will not explain the experimental results, whatever be the refractive index for the pressural-wave.

(2.) Lord KELVIN's contractile ether theory and the electromagnetic theory in HERTZ's form, lead to the same equations, containing three independent constants (of which two have little effect, except at a distance from the polarizing angle); and these equations agree with the experiment rather better than CAUCHY's empirical formulæ containing, as used by JAMIN, one constant, ϵ, and as used by QUINCKE, two constants, ϵ and μ'. At a distance from the polarizing angle FRESNEL's expression for the intensity is sufficient.

(3.) Whilst CAUCHY's constants, ϵ, have been found not to satisfy the theoretical conditions assigned by JAMIN (so that CAUCHY's formula must be regarded as an empirical one), the constants of the above theoretical formulæ satisfy the conditions theoretically deduced, as nearly as is to be expected, considering that the whole effect under discussion is itself but a small correction.

This last conclusion as to the possibility of a theoretical explanation of the phenomena of reflection based on the existence of a film of transition is at variance with the result arrived at by M. H. BOUASSE from a critical examination of the theories so far proposed. (See his paper in the 'Annales de Chimie et Physique' for February, 1893, p. 145.)

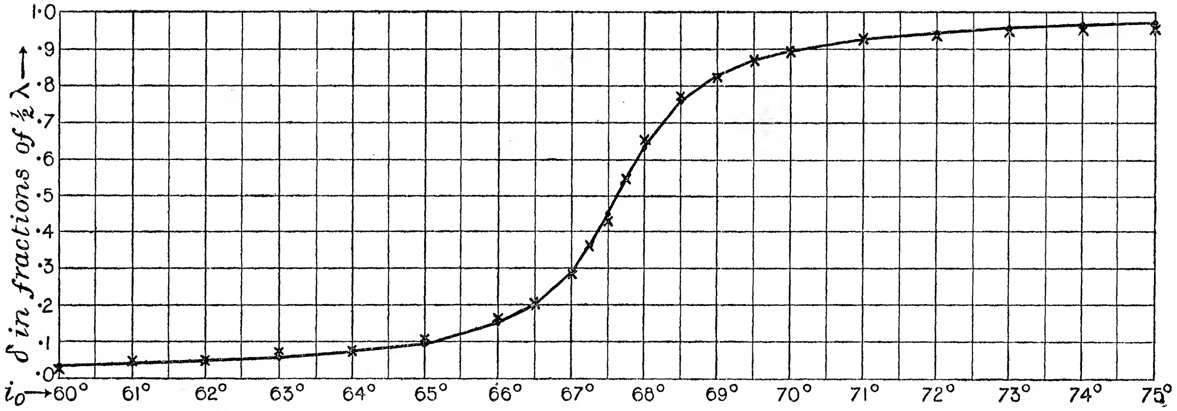

Diagram of the difference of phase of the components of Light reflected in Air from Diamond (according to Jamin's experiments).

The black line is the theoretical phase curve; the crosses represent Jamin's experimental results.

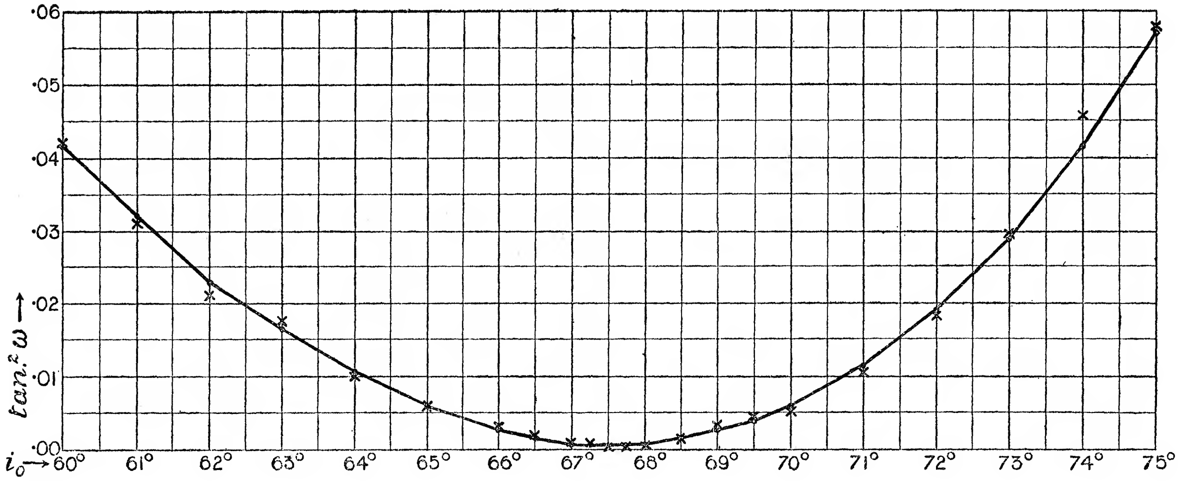

Diagram of ratio of intensities of Light reflected in Air from Diamond (according to Jamin's experiments).

The black line is the theoretical curve; the crosses show Jamin's experimental results.